# Deriving Upper and Lower Bounds of Hausdorff Distance for Polygonal Models

Min Tang and Young J. Kim

Dept of Computer Science and Engineering
Ewha Womans Univ., Seoul, Korea
{*tangmin, kimy*}@*ewha.ac.kr*

In this document, we prove the lemmas and theorems presented in the paper, *"Interactive Hausdorff Distance Computation for Polygonal Models"*.

**DEFINITION 1**
*Given two compact sets $\mathcal{A}, \mathcal{B}$ in $\mathbb{R}^3$, the one-sided Hausdorff distance*[1] *from $\mathcal{A}$ to $\mathcal{B}$ is defined as:*

$$h(\mathcal{A}, \mathcal{B}) \equiv \max_{\mathbf{a} \in \mathcal{A}} \min_{\mathbf{b} \in \mathcal{B}} d(\mathbf{a}, \mathbf{b}), \tag{1}$$

*where $d(\cdot, \cdot)$ denotes the Euclidean distance operator in $\mathbb{R}^3$. Then, the two-sided Hausdorff distance between $\mathcal{A}$ and $\mathcal{B}$ is defined as:*

$$H(\mathcal{A}, \mathcal{B}) \equiv \max(h(\mathcal{A}, \mathcal{B}), h(\mathcal{B}, \mathcal{A})). \tag{2}$$

*From the above definition, one can derive the following theorem for polygonal models.*

**THEOREM 1**
*If $\mathcal{A}, \mathcal{B}$ are polygonal models, and $\Delta^{\mathcal{A}}$ denotes a triangle in $\mathcal{A}$, then*

$$h(\mathcal{A}, \mathcal{B}) = \max_{\Delta^{\mathcal{A}} \in \mathcal{A}} \left( h(\Delta^{\mathcal{A}}, \mathcal{B}) \right) \tag{3}$$

*From Theorem 1, computing $h(\mathcal{A}, \mathcal{B})$ boils down to computing $h(\Delta^{\mathcal{A}}, \mathcal{B})$. Now we show lemmas related to bounds of Hausdorff distance metric.*

**LEMMA 1**
*Given compact sets $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ with $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B} \subseteq \mathcal{B}'$, the following inequalities hold:*

$$\begin{aligned} h(\mathcal{A}, \mathcal{B}') &\leq h(\mathcal{A}, \mathcal{B}) \\ h(\mathcal{A}, \mathcal{B}) &\leq h(\mathcal{A}', \mathcal{B}) \end{aligned} \tag{4}$$

[1] Whenever distinguishable from the context, we simply refer to Hausdorff distance as one-sided Hausdroff distance throughout the paper.

***Proof*** *Since $\mathcal{B} \subseteq \mathcal{B}'$ and $\mathcal{B}' - \mathcal{B} \neq \emptyset$,*

$$\begin{aligned} & h(\mathcal{A}, \mathcal{B}') \\ & = \max_{\mathbf{p} \in \mathcal{A}} (d(\mathbf{p}, \mathcal{B}')) \\ & = \max_{\mathbf{p} \in \mathcal{A}} (\min(d(\mathbf{p}, \mathcal{B}), d(\mathbf{p}, \mathcal{B}' - \mathcal{B}))) \\ & \leq \max_{\mathbf{p} \in \mathcal{A}} (d(\mathbf{p}, \mathcal{B})) \\ & = h(\mathcal{A}, \mathcal{B}) \end{aligned}$$

*Let $\mathbf{q} \in \mathcal{A}$ be a point realizing $d(\mathbf{q}, \mathcal{B}) = h(\mathcal{A}, \mathcal{B})$. Since $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathbf{q} \in \mathcal{A}'$,*

$$h(\mathcal{A}', \mathcal{B}) = \max_{\mathbf{p} \in \mathcal{A}'} (d(\mathbf{p}, \mathcal{B})) \geq d(\mathbf{q}, \mathcal{B}) = h(\mathcal{A}, \mathcal{B})$$

□

*Based on the above lemma, we present a simple way to compute upper and lower bounds of the one-sided Hausdorff distance between polygonal models.*

**LEMMA 2**
*Let $\mathbf{v}_i^{\mathcal{A}} (i = 1, 2, 3)$ represent one of the three vertices in a triangle $\Delta^{\mathcal{A}} \in \mathcal{A}$. Then, upper and lower bounds of $h\left(\Delta^{\mathcal{A}}, B\right)$ can be obtained as:*

$$\begin{aligned} \overline{h}(\Delta^{\mathcal{A}}, \mathcal{B}) &= \min_{\Delta_{\mathcal{B}} \in \mathcal{B}} \left(h(\Delta^{\mathcal{A}}, \Delta^{\mathcal{B}})\right) \\ \underline{h}(\Delta^{\mathcal{A}}, \mathcal{B}) &= \max_{i=1,2,3} \left(d(\mathbf{v}_i^{\mathcal{A}}, \mathcal{B})\right) \end{aligned} \tag{5}$$

***Proof*** *Since $\mathbf{v}_i \in \mathcal{A}$ and $h(\mathbf{v}_i^{\mathcal{A}}, \mathcal{B}) = d(\mathbf{v}_i^{\mathcal{A}}, \mathcal{B})$, we have $d(\mathbf{v}_i^{\mathcal{A}}, \mathcal{B}) \leq h(\Delta^{\mathcal{A}}, \mathcal{B})$ for all $i$. Thus, $\max_i (d(\mathbf{v}_i^{\mathcal{A}}, \mathcal{B})) \leq h(\Delta^{\mathcal{A}}, \mathcal{B})$. Moreover, for all $\Delta^{\mathcal{B}} \in \mathcal{B}$, we have $h(\Delta^{\mathcal{A}}, \mathcal{B}) \leq h(\Delta^{\mathcal{A}}, \Delta^{\mathcal{B}})$. Thus, $h(\Delta^{\mathcal{A}}, \mathcal{B}) \leq \min_{\Delta_{\mathcal{B}} \in \mathcal{B}} (h(\Delta^{\mathcal{A}}, \Delta^{\mathcal{B}}))$.*
□

**THEOREM 2**
*We have the upper bound $\overline{h}$ and lower bound $\underline{h}$ of $h(\mathcal{A}, \mathcal{B})$ as:*

$$\begin{aligned} \overline{h}(\mathcal{A}, \mathcal{B}) &= \max_{\Delta^{\mathcal{A}} \in \mathcal{A}} \left(\overline{h}(\Delta^{\mathcal{A}}, \mathcal{B})\right) \\ \underline{h}(\mathcal{A}, \mathcal{B}) &= \max_{\Delta^{\mathcal{A}} \in \mathcal{A}} \left(\underline{h}(\Delta^{\mathcal{A}}, \mathcal{B})\right) \end{aligned} \tag{6}$$

***Proof*** *The result follows from Lemma 2.* □

*Now we prove the lemmas for the Voronoi subdivision method.*

**LEMMA 3**
*Given the subdivided triangle $\Delta_s^{\mathcal{A}} \subseteq \Delta^{\mathcal{A}}$, if $\overline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) = d(\mathbf{v}_u^{\mathcal{A}}, \Delta_u^{\mathcal{B}})$ and $\underline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) = d(\mathbf{v}_l^{\mathcal{A}}, \Delta_l^{\mathcal{B}})$ for some $\mathbf{v}_u^{\mathcal{A}}, \mathbf{v}_l^{\mathcal{A}} \in \Delta_s^{\mathcal{A}}$, then $\{\Delta_u^{\mathcal{B}}, \Delta_l^{\mathcal{B}}\} \subseteq \mathtt{L}$ where $\mathtt{L}$ is obtained in the step 1 of subdivision.*

***Proof*** *Suppose $\Delta_u^{\mathcal{B}} \notin \mathtt{L}$, then $d\left(\Delta^{\mathcal{A}}, \Delta_u^{\mathcal{B}}\right) > \overline{h}(\Delta^{\mathcal{A}}, \mathcal{B})$ by the definition of $\mathtt{L}$. Further, $\overline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) = d(\mathbf{v}_u^{\mathcal{A}}, \Delta_u^{\mathcal{B}}) \geq d\left(\Delta_s^{\mathcal{A}}, \Delta_u^{\mathcal{B}}\right)$ since $\mathbf{v}_u^{\mathcal{A}} \in \Delta_s^{\mathcal{A}}$, and $d\left(\Delta_s^{\mathcal{A}}, \Delta_u^{\mathcal{B}}\right) \geq d\left(\Delta^{\mathcal{A}}, \Delta_u^{\mathcal{B}}\right)$ since $\Delta_s^{\mathcal{A}} \subseteq \Delta^{\mathcal{A}}$. Thus, we have $\overline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) > \overline{h}(\Delta^{\mathcal{A}}, \mathcal{B})$. However, since $\Delta_s^{\mathcal{A}} \subseteq \Delta^{\mathcal{A}}$, $h\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) \leq h\left(\Delta^{\mathcal{A}}, \mathcal{B}\right)$. Moreover since $\min_{\Delta_{\mathcal{B}} \in \mathcal{B}} \left(h(\Delta_s^{\mathcal{A}}, \Delta^{\mathcal{B}})\right) \leq \min_{\Delta_{\mathcal{B}} \in \mathcal{B}} \left(h(\Delta^{\mathcal{A}}, \Delta^{\mathcal{B}})\right)$*

*and* $\overline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) \equiv \min_{\Delta_{\mathcal{B}} \in \mathcal{B}} \left(h(\Delta_s^{\mathcal{A}}, \Delta^{\mathcal{B}})\right), \overline{h}\left(\Delta^{\mathcal{A}}, \mathcal{B}\right) \equiv \min_{\Delta_{\mathcal{B}} \in \mathcal{B}} \left(h(\Delta^{\mathcal{A}}, \Delta^{\mathcal{B}})\right)$ *using Lemma 2,* $\overline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) \leq \overline{h}(\Delta^{\mathcal{A}}, \mathcal{B})$. *We have contradiction.*

*Suppose* $\Delta_l^{\mathcal{B}} \notin \mathtt{L}$*, then* $d\left(\Delta^{\mathcal{A}}, \Delta_l^{\mathcal{B}}\right) > \overline{h}(\Delta^{\mathcal{A}}, \mathcal{B})$ *by the definition of* $\mathtt{L}$*. Further,* $\underline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) = d(\mathbf{v}_l^{\mathcal{A}}, \Delta_l^{\mathcal{B}}) \geq d\left(\Delta_s^{\mathcal{A}}, \Delta_l^{\mathcal{B}}\right)$ *and thus* $\underline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) > \overline{h}(\Delta^{\mathcal{A}}, \mathcal{B})$*, which is contradiction since* $\underline{h}\left(\Delta_s^{\mathcal{A}}, \mathcal{B}\right) \leq h(\Delta_s^{\mathcal{A}}, \mathcal{B}) \leq h(\Delta^{\mathcal{A}}, \mathcal{B}) \leq \overline{h}(\Delta^{\mathcal{A}}, \mathcal{B})$.
□

**LEMMA 4**
*Let* $\mathbf{v}^{\mathcal{A}}, \Delta^{\mathcal{B}}$ *be the vertex in* $\mathcal{A}$ *and triangle in* $\mathcal{B}$ *that realize* $\underline{h}(\Delta^{\mathcal{A}}, \mathcal{B})$*; i.e.* $\underline{h}(\Delta^{\mathcal{A}}, \mathcal{B}) = d(\mathbf{v}^{\mathcal{A}}, \mathcal{B})$*. Further, let us call the closest direction vector from* $\mathbf{v}^{\mathcal{A}}$ *to* $\Delta^{\mathcal{B}}$ *as* $\mathbf{d}$ *such that* $||\mathbf{d}|| = d(\mathbf{v}^{\mathcal{A}}, \mathcal{B})$*. Then, when we translate* $\Delta^{\mathcal{A}}$ *along* $\mathbf{d}$ *by* $||\mathbf{d}|| + \varepsilon$*, if* $\Delta^{\mathcal{A}}$ *is completely enclosed by the model* $\mathcal{B}$*, the following inequalities hold:*

$$\forall \mathbf{p} \in \Delta_{\mathcal{A}},\ d\left(\mathbf{p}, \mathcal{B}\right) \leq \underline{h}\left(\Delta^{\mathcal{A}}, \mathcal{B}\right) + \varepsilon \tag{7}$$

*Thus,* $h\left(\Delta^{\mathcal{A}}, \mathcal{B}\right) = \max_{\mathbf{p} \in \Delta_{\mathcal{A}}} d\left(\mathbf{p}, \mathcal{B}\right) \leq \underline{h}\left(\Delta^{\mathcal{A}}, \mathcal{B}\right) + \varepsilon$

***Proof*** *As illustrated in Fig.1, let* $\Delta^{\mathcal{A}'}$ *be an affine copy of the triangle* $\Delta^{\mathcal{A}}$ *after being translated along* $\mathbf{d}$ *by* $||\mathbf{d}|| + \varepsilon$*. Further, let* $\mathbf{p}$ *be any point on* $\Delta^{\mathcal{A}}$ *and* $\mathbf{p}'$ *be the corresponding point on* $\Delta^{\mathcal{A}'}$*. If we denote* $\mathbf{q}$ *as a point on* $\mathcal{B}$ *intersected with the line segment* $\mathbf{pp}'$*, the following inequality should be satisfied:*

$$d\left(\mathbf{p}, \mathcal{B}\right) \leq d(\mathbf{pq}) < d(\mathbf{pp}') = \underline{h}\left(\Delta^{\mathcal{A}}, \mathcal{B}\right) + \varepsilon$$

□

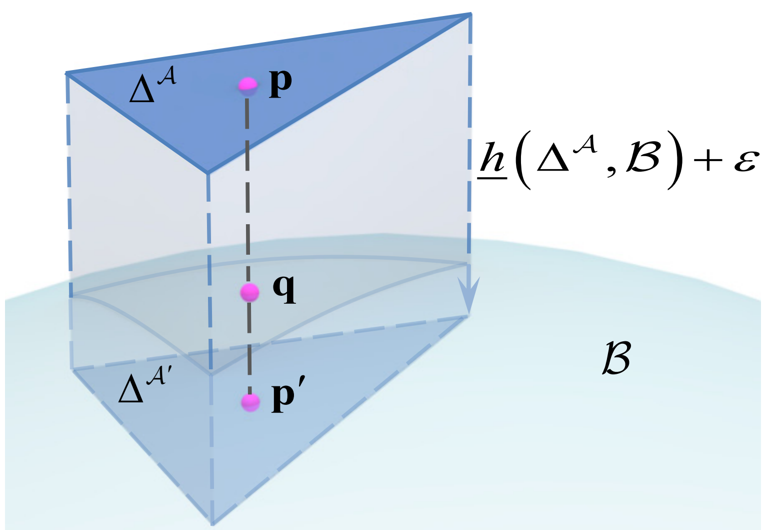


Figure 1: ***Termination Condition for Closed Models.*** $\Delta^{\mathcal{A}}$ *is enclosed by* $\mathcal{B}$ *when* $\Delta^{\mathcal{A}}$ *is translated by* $\underline{h}(\Delta^{\mathcal{A}}, \mathcal{B}) + \varepsilon$.