Penetration Depth Calculation for Tetrahedra using SAT

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In this document, we prove Eq. 11 in the paper, *"A Penetration Metric for Deforming Tetrahedra using Object Norm"* [Kim and Kim 2019].

Definition 1. *The projected length $L_{\mathbf{n}}(\mathcal{T})$ of a given simplicial complex $\mathcal{T}$ along an axial direction $\mathbf{n}$ is defined as:*

$$L_{\mathbf{n}}(\mathcal{T}) = \max(\{|(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{n}| \mid \forall \mathbf{x}_i, \mathbf{x}_j \in \mathcal{T}\}) \tag{1}$$

where $\mathbf{x}_i, \mathbf{x}_j$ are the vertices in $\mathcal{T}$. Then, the pair of vertices $\mathbf{x}_i, \mathbf{x}_j$ that realizes the projected length $L_{\mathbf{n}}(\mathcal{T})$ is called supporting vertices for $\mathbf{n}$.

Theorem 1. *The penetration depth (PD) of two intersected tetrahedra $\mathcal{T}_1, \mathcal{T}_2$ can be calculated as:*

$$\mathrm{PD}(\mathcal{T}_1, \mathcal{T}_2) = \min\{(L_{\mathbf{n}}(\mathcal{T}_1) + L_{\mathbf{n}}(\mathcal{T}_2) - L_{\mathbf{n}}(\mathcal{T}_1 \cup \mathcal{T}_2)) \mid \forall \mathbf{n} \in N\}, \tag{2}$$

where $N = \{\mathbf{n}_{\mathrm{FV}}, \mathbf{n}_{\mathrm{VF}}, \mathbf{n}_{\mathrm{EE}}\}$ is a set of possible separating directions for $\mathcal{T}_1, \mathcal{T}_2$.

Proof. According to the separating axis theorem (SAT) [Gottschalk et al. 1996], two convex objects $\mathcal{T}_1, \mathcal{T}_2$ do not overlap iff there exists a separating axis $\mathbf{n}$ that the axial projection of $\mathcal{T}_1, \mathcal{T}_2$ onto $\mathbf{n}$ does not overlap. The SAT can be rewritten using Eq. 1 as:

$$\exists \mathbf{n} \in N,\ L_{\mathbf{n}}(\mathcal{T}_1 \cup \mathcal{T}_2) \geq L_{\mathbf{n}}(\mathcal{T}_1) + L_{\mathbf{n}}(\mathcal{T}_2). \tag{3}$$

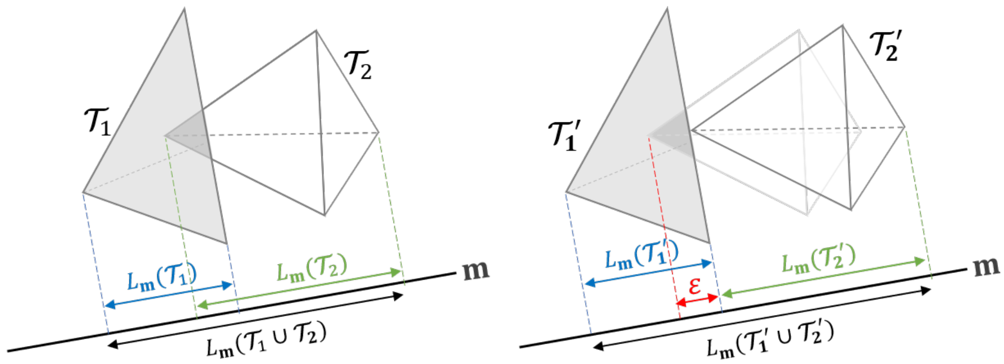
Thus, if two objects are interpenetrated, $L_{\mathbf{n}}(\mathcal{T}_1) + L_{\mathbf{n}}(\mathcal{T}_2) - L_{\mathbf{n}}(\mathcal{T}_1 \cup \mathcal{T}_2) > 0$ for $\forall \mathbf{n}$. Let ε be the result of evaluating Eq. 2 and $\mathbf{m}$ be the corresponding separating direction: i.e.,

$$\mathbf{m} = \underset{\mathbf{n}}{\mathrm{argmin}}\{(L_{\mathbf{n}}(\mathcal{T}_1) + L_{\mathbf{n}}(\mathcal{T}_2) - L_{\mathbf{n}}(\mathcal{T}_1 \cup \mathcal{T}_2)) \mid \forall \mathbf{n} \in N\} \tag{4}$$

Then, we can prove Theorem 1 by showing that:

1. $\varepsilon\mathbf{m}$ separates $\mathcal{T}_1$ and $\mathcal{T}_2$ by translation.

2. ε is the smallest magnitude among such translations.



(a) Interpenetrated state (b) Translated state

Figure 1: *Example of two tetrahedra projected on the axis* $\mathbf{m}$ *before and after translation* $\varepsilon\mathbf{m}$*, where* $\mathbf{m}$ *is the face normal of* $\mathcal{T}_1$*. The projected length of each tetrahedron does not change during translation.*

Let $\mathcal{T}_1', \mathcal{T}_2'$ be the tetrahedra of $\mathcal{T}_1, \mathcal{T}_2$ translated by $\varepsilon\mathbf{m}$. Since these tetrahedra are not rotated, the projected length of each tetrahedron on axis $\mathbf{m}$ is the same as before translation:

$$
\begin{aligned}
L_\mathbf{m}(\mathcal{T}_1) &= L_\mathbf{m}(\mathcal{T}_1') \\
L_\mathbf{m}(\mathcal{T}_2) &= L_\mathbf{m}(\mathcal{T}_2')
\end{aligned}
\tag{5}
$$

Since the direction of the translation is the same as the projection axis, the length of the translation vector is $|\varepsilon\mathbf{m} \cdot \mathbf{m}| = \varepsilon$. Then, the entire projected length of the translated tetrahedra is:

$$
\begin{aligned}
L_\mathbf{m}(\mathcal{T}_1' \cup \mathcal{T}_2') &= L_\mathbf{m}(\mathcal{T}_1 \cup \mathcal{T}_2) + \varepsilon \\
&= L_\mathbf{m}(\mathcal{T}_1 \cup \mathcal{T}_2) + L_\mathbf{m}(\mathcal{T}_1) + L_\mathbf{m}(\mathcal{T}_2) - L_\mathbf{m}(\mathcal{T}_1 \cup \mathcal{T}_2) \\
&= L_\mathbf{m}(\mathcal{T}_1) + L_\mathbf{m}(\mathcal{T}_2) \\
&= L_\mathbf{m}(\mathcal{T}_1') + L_\mathbf{m}(\mathcal{T}_2'),
\end{aligned}
\tag{6}
$$

implying that the two tetrahedra translated by $\varepsilon\mathbf{m}$ do not overlap because of the SAT (Fig. 1(b)).

Let $\tilde{\varepsilon}\tilde{\mathbf{m}}$ be an arbitrary translation that separates $\mathcal{T}_1, \mathcal{T}_2$ and $\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2$ be the translated copies of the tetrahedra. Since $\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2$ are separated, according to the SAT, there exists a separating axis $\tilde{\mathbf{n}}$ that satisfies $L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}}_1 \cup \tilde{\mathcal{T}}_2) \geq$

$L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_1}) + L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_2})$. For the given set $\mathcal{T}_1 \cup \mathcal{T}_2$ and the direction $\tilde{\mathbf{n}}$, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the two supporting vertices used to calculate the projection $L_{\tilde{\mathbf{n}}}(\mathcal{T}_1 \cup \mathcal{T}_2) = |(\mathbf{x}_2 - \mathbf{x}_1) \cdot \tilde{\mathbf{n}}|$.

Now, suppose that these two vertices can support the tetrahedra even after being translated by $\tilde{\varepsilon}\tilde{\mathbf{m}}$ (Fig. 2(a)), and let $\tilde{\mathbf{x}}_\mathbf{1}$ and $\tilde{\mathbf{x}}_\mathbf{2}$ be the corresponding vertices after translation. Then the displacement between the two vertices after translation can be calculated as: $\tilde{\mathbf{x}}_\mathbf{2} - \tilde{\mathbf{x}}_\mathbf{1} = \mathbf{x}_2 - \mathbf{x}_1 + \varepsilon\tilde{\mathbf{m}}$. The projected length is $L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_1} \cup \tilde{\mathcal{T}_2}) = |(\tilde{\mathbf{x}}_\mathbf{2} - \tilde{\mathbf{x}}_\mathbf{1}) \cdot \tilde{\mathbf{n}}| = |(\mathbf{x}_2 - \mathbf{x}_1 + \varepsilon\tilde{\mathbf{m}}) \cdot \tilde{\mathbf{n}}|$. Then,

$$\begin{aligned} L_{\tilde{\mathbf{n}}}(\mathcal{T}_1 \cup \mathcal{T}_2) + |\tilde{\varepsilon}\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| &= |(\mathbf{x}_2 - \mathbf{x}_1) \cdot \tilde{\mathbf{n}}| + |\varepsilon\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| \\ &\geq |(\mathbf{x}_2 - \mathbf{x}_1) \cdot \tilde{\mathbf{n}} + \varepsilon\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| = L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_1} \cup \tilde{\mathcal{T}_2}) \\ &\geq L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_1}) + L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_2}) = L_{\tilde{\mathbf{n}}}(\mathcal{T}_1) + L_{\tilde{\mathbf{n}}}(\mathcal{T}_2) \end{aligned} \tag{7}$$

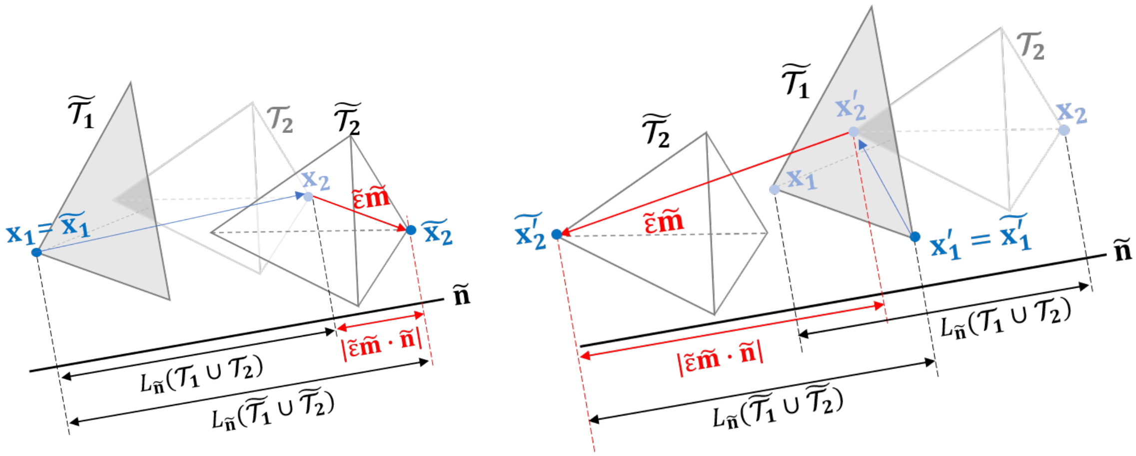
Otherwise, the supporting vertices are changed after translation (Fig. 2(b)). Let $\tilde{\mathbf{x}_1}'$, $\tilde{\mathbf{x}_2}'$ be the supporting vertices after translation and $\mathbf{x}_1'$, $\mathbf{x}_2'$ be the corresponding vertices before translation. Then, $L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_1} \cup \tilde{\mathcal{T}_2}) = |(\tilde{\mathbf{x}}_\mathbf{2}' - \tilde{\mathbf{x}}_\mathbf{1}') \cdot \tilde{\mathbf{n}}| = |(\mathbf{x}_2' - \mathbf{x}_1' + \varepsilon\tilde{\mathbf{m}}) \cdot \tilde{\mathbf{n}}|$. Since $\mathbf{x}_1'$ and $\mathbf{x}_2'$ were not supporting vertices before translation, according to Eq. 1, the projected length of $\mathbf{x}_1'$ and $\mathbf{x}_2'$ must be smaller than that of supporting vertices $\mathbf{x}_1$, $\mathbf{x}_2$: i.e., $L_{\tilde{\mathbf{n}}}(\mathcal{T}_1 \cup \mathcal{T}_2) = |(\mathbf{x}_2 - \mathbf{x}_1) \cdot \tilde{\mathbf{n}}| \geq |(\mathbf{x}_2' - \mathbf{x}_1') \cdot \tilde{\mathbf{n}}|$. Thus,

$$\begin{aligned} L_{\tilde{\mathbf{n}}}(\mathcal{T}_1 \cup \mathcal{T}_2) + |\tilde{\varepsilon}\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| &= |(\mathbf{x}_2 - \mathbf{x}_1) \cdot \tilde{\mathbf{n}}| + |\varepsilon\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| \\ &\geq |(\mathbf{x}_2' - \mathbf{x}_1') \cdot \tilde{\mathbf{n}}| + |\varepsilon\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| \\ &\geq |(\mathbf{x}_2' - \mathbf{x}_1') \cdot \tilde{\mathbf{n}} + \varepsilon\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| = L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_1} \cup \tilde{\mathcal{T}_2}) \\ &\geq L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_1}) + L_{\tilde{\mathbf{n}}}(\tilde{\mathcal{T}_2}) = L_{\tilde{\mathbf{n}}}(\mathcal{T}_1) + L_{\tilde{\mathbf{n}}}(\mathcal{T}_2) \end{aligned} \tag{8}$$

In either case, we can see that $|\tilde{\varepsilon}\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| \geq L_{\tilde{\mathbf{n}}}(\mathcal{T}_1) + L_{\tilde{\mathbf{n}}}(\mathcal{T}_2) - L_{\tilde{\mathbf{n}}}(\mathcal{T}_1 \cup \mathcal{T}_2)$. Since $|\tilde{m}| = |\tilde{n}| = 1$,

$$\begin{aligned} \tilde{\varepsilon} &\geq \tilde{\varepsilon}|\tilde{\mathbf{m}} \cdot \tilde{\mathbf{n}}| \\ &\geq L_{\tilde{\mathbf{n}}}(\mathcal{T}_1) + L_{\tilde{\mathbf{n}}}(\mathcal{T}_2) - L_{\tilde{\mathbf{n}}}(\mathcal{T}_1 \cup \mathcal{T}_2) \\ &\geq \varepsilon \end{aligned} \tag{9}$$

Therefore, ε is the minimum translational distance that separates $\mathcal{T}_1, \mathcal{T}_2$. $\square$



(a) Supporting vertices are same

(b) Supporting vertices are changed

Figure 2: *Examples of an arbitrary translation $\tilde{\varepsilon}\tilde{\mathbf{m}}$ that separates two tetrahedra and their projection on the separating axis $\tilde{\mathbf{n}}$. (a)(b) shows two cases of projection results according to the changes in supporting vertices before and after translation.*

References

GOTTSCHALK, S., LIN, M. C., AND MANOCHA, D. 1996. Obbtree: A hierarchical structure for rapid interference detection. In *Proceedings of the 23rd annual conference on Computer graphics and interactive techniques*, ACM, 171–180.

KIM, J., AND KIM, Y. J. 2019. A penetration metric for deforming tetrahedra using object norm. In *2019 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, IEEE. Under Review.