Motion Interpolation and Bounds Calculations in CATCH

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In this document, for a link $\mathscr{A}_i$, we explain in detail how to synthesize its interpolating motion ${}^0_i\widehat{\mathbf{M}}(t)$ between two configurations $\mathbf{q}_0, \mathbf{q}_1$ and derive an upper bound of motion (Eq. 10 of the main paper for CATCH).

1 Motion Interpolation

Let us start by describing the motion ${}^{i-1}_i\widehat{\mathbf{M}}$ of $\{i\}$ relative to the reference frame of its unique parent $\{i-1\}$. We use a 3×1 matrix ${}^{i-1}_i\mathbf{T}$ and a 3×3 matrix ${}^{i-1}_i\mathbf{R}$ to denote, respectively, the position and orientation of $\{i\}$ relative to $\{i-1\}$ at the beginning of the time interval $[0,1]$. Then, we can represent ${}^{i-1}_i\widehat{\mathbf{M}}$ with a rotation around an axis ${}^{i-1}\mathbf{u}_i$ by an angle ${}^{i-1}\theta_i$ and a translation through ${}^{i-1}\mathbf{v}_i$. Moreover, ${}^{i-1}_i\mathbf{T}$ and ${}^{i-1}_i\mathbf{R}$ can be determined from the relative configuration of $\{i\}$ with respect to $\{i-1\}$, and ${}^{i-1}\theta_i$, ${}^{i-1}\mathbf{u}_i$, ${}^{i-1}\mathbf{v}_i$ from the relative motion of $\{i\}$ with respect to $\{i-1\}$. Thus, for a given time interval $[0,1]$, ${}^{i-1}_i\mathbf{T}$, ${}^{i-1}_i\mathbf{R}$, ${}^{i-1}\mathbf{u}_i$ and ${}^{i-1}\mathbf{v}_i$ are all constants and can all be expressed in terms of $\{i-1\}$. Moreover, we linearly interpolate $\mathbf{q}_0$ and $\mathbf{q}_1$ while $\{i\}$ moves with constant translational and rotational velocities with respect to $\{i-1\}$, so that ${}^{i-1}\theta_i$ also becomes constant. Note that the rotational velocity, ${}^{i-1}\omega_i = {}^{i-1}\theta_i\,{}^{i-1}\mathbf{u}_i$.

The position of $\{i\}$ relatively to $\{i-1\}$ at a given time t in $[0,1]$ can now be written

$$ {}^{i-1}_i\widehat{\mathbf{T}}(t) = {}^{i-1}_i\mathbf{T} + {}^{i-1}\mathbf{v}_i\, t, \tag{1} $$

and the orientation of $\{i\}$ relative to $\{i-1\}$ is expressed as

$$ {}^{i-1}_i\widehat{\mathbf{R}}(t) = \cos({}^{i-1}\theta_i\, t)\cdot{}^{i-1}_i\mathbf{A} + \sin({}^{i-1}\theta_i\, t)\cdot{}^{i-1}_i\mathbf{B} + {}^{i-1}_i\mathbf{C}. \tag{2} $$

${}^{i-1}_i\mathbf{A}$, ${}^{i-1}_i\mathbf{B}$ and ${}^{i-1}_i\mathbf{C}$ are 3×3 constant matrices which are computed at the beginning of the time interval $[0,1]$:

$$ \begin{aligned} {}^{i-1}_i\mathbf{A} &= \mathbf{R}_i - \mathbf{u}_i\cdot\mathbf{u}_i^T\cdot\mathbf{R}_i \\ {}^{i-1}_i\mathbf{B} &= \mathbf{u}_i^*\cdot\mathbf{R}_i \\ {}^{i-1}_i\mathbf{C} &= \mathbf{u}_i\cdot\mathbf{u}_i^T\cdot\mathbf{R}_i. \end{aligned} \tag{3} $$

$\mathbf{R}_i$ is also computed at the beginning of $[0,1]$ and $\mathbf{R}_i = {}^{i-1}_i\widehat{\mathbf{R}}(1){}^{i-1}_i\widehat{\mathbf{R}}(0)^T$. Here, ${}^{i-1}_i\widehat{\mathbf{R}}(1)$ and ${}^{i-1}_i\widehat{\mathbf{R}}(0)$ are the orientations of $\{i\}$ relative to $\{i-1\}$ at, respectively, the beginning and end of the time interval $[0,1]$. $\mathbf{u}_i^*$ denote the 3×3 matrix such as $\mathbf{u}_i^*\mathbf{x} = \mathbf{u}_i \times \mathbf{x}$ for every three-dimensional vector $\mathbf{x}$. If $\mathbf{u}_i = (u_i^x, u_i^y, u_i^z)^T$, then:

$$ \mathbf{u}_i^* = \begin{pmatrix} 0 & -u_i^z & u_i^y \\ u_i^z & 0 & -u_i^x \\ -u_i^y & u_i^x & 0 \end{pmatrix} \tag{4} $$

Consequently, the motion of $\{i\}$ relatively to $\{i-1\}$ is described by the 4×4 homogeneous matrix

$$ {}^{i-1}_{i}\widehat{\mathbf{M}}(t) = \begin{pmatrix} {}^{i-1}_{i}\widehat{\mathbf{R}}(t) & {}^{i-1}_{i}\widehat{\mathbf{T}}(t) \\ (0,0,0) & 1 \end{pmatrix} \tag{5} $$

in the reference frame of $\{i-1\}$. Finally, the matrix:

$$ {}^{0}_{i}\widehat{\mathbf{M}}(t) = {}^{0}_{1}\widehat{\mathbf{M}}(t)\cdot{}^{1}_{2}\widehat{\mathbf{M}}(t)\ldots{}^{i-1}_{i}\widehat{\mathbf{M}}(t) \tag{6} $$

describes the motion of link i in the world reference frame $\{0\}$.

Note that this formulation makes it extremely simple to compute all the motion parameters ${}^{i-1}\mathbf{v}_i$ and ${}^{i-1}\omega_i$ at a given time t. For a given link i, we assume that ${}^{i-1}_{i}\widehat{\mathbf{T}}(0)$ and ${}^{i-1}_{i}\widehat{\mathbf{T}}(1)$ are the initial and final positions of $\{i\}$ relative to $\{i-1\}$ and ${}^{i-1}_{i}\widehat{\mathbf{R}}(0)$ and ${}^{i-1}_{i}\widehat{\mathbf{R}}(1)$ are the initial and final orientations relative to $\{i-1\}$. Then ${}^{i-1}\mathbf{v}_i = {}^{i-1}_{i}\widehat{\mathbf{T}}(1) - {}^{i-1}_{i}\widehat{\mathbf{T}}(0)$, and ${}^{i-1}\omega_i$ represent the rotation extracted from the rotation matrix ${}^{i-1}_{i}\widehat{\mathbf{R}}(1)\cdot{}^{i-1}_{i}\widehat{\mathbf{R}}(0)^T$.

2 Derivation of Motion Bound

The upper bound of motion for link i is obtained by maximizing the *projected* trajectory length over all points on the link when the link undergoes a *linearly interpolating motion* with constant translational and rotational velocities. Let us call ${}^{0}\mathbf{p}_i$ an arbitrary point on link i. By projecting its velocity ${}^{0}\mathbf{v}_i$ onto the closest direction vector $\mathbf{n}$ between link i and its potentially colliding objects, integrating it over the time interval $[0,1]$ (i.e., projected trajectory length), and maximizing the trajectory length for all ${}^{0}\mathbf{p}_i$ on link i, we obtain an upper bound of motion for link i as shown in Eq. 9 and subsequently Eq. 10 by using triangular inequality.

In Eq. 10, notice that $\max\limits_{{}^{0}_{i}\mathbf{p}\in\mathscr{A}_i}()$ depends on only ${}^{j-1}\mathbf{L}_j$ of the current link ($j = i$) and is independent of its ancestor links ($j < i$). Therefore, in order to eliminate $\max\limits_{{}^{0}_{i}\mathbf{p}\in\mathscr{A}_i}()$, we consider the following two cases:

1. $j = i$. ${}^{j-1}\mathbf{L}_j$ is a vector from a point on link i to its rotational center: ${}^{j-1}_{j}\mathbf{r}_{\mathbf{p}}(t)$. The rotational center is the origin of frame $\{i-1\}$ in our case. The length of the vector $\left|{}^{j-1}_{j}\mathbf{r}_{\mathbf{p}}(t)\right|$ is independent of time t (see Fig. 1-(b)), so an upper bound of $\left|{}^{j-1}\mathbf{L}_j(t)\right|$ can be written as follows:

$$ \left|{}^{j-1}\mathbf{L}_j(t)\right| \le \max_{{}^{0}_{i}\mathbf{p}\in\mathscr{A}_i}\left(\left|{}^{j-1}\mathbf{r}_l\right|\right) = \left|{}^{j-1}\mathbf{L}_j(t)\right|_{\mu} \tag{7} $$

2. $j < i$. The link j is an ancestor of the link i. Recall that ${}^{j-1}\mathbf{L}_j$ is a vector from the origin of link i to the origin of frame $\{i-1\}$ in this case (also see Fig. 1-(c)). Since ${}^{j-1}\mathbf{v}_j$ is a constant, ${}^{j-1}\mathbf{L}_j(t)$ can be simply defined as :

$$ {}^{j-1}\mathbf{L}_j(t) = {}^{j-1}\mathbf{L}_j(0) + {}^{j-1}\mathbf{v}_j\, t $$

Obviously,

$$ \left|{}^{j-1}\mathbf{L}_j(t)\right| \le \max\left(\left|{}^{j-1}\mathbf{L}_j(0)\right|, \left|{}^{j-1}\mathbf{L}_j(1)\right|\right) = \left|{}^{j-1}\mathbf{L}_j(t)\right|_{\mu} \tag{8} $$

Particularly if $\{j\}$ has no translational motion relative to $\{j-1\}$ (i.e., $\left|{}^{j-1}\mathbf{v}_j\right| = 0$), then $\left|{}^{j-1}\mathbf{L}_j\right|$ is a constant which results in $\left|{}^{j-1}\mathbf{L}_j(t)\right| = \left|{}^{j-1}\mathbf{L}_j(0)\right| = \left|{}^{j-1}\mathbf{L}_j(t)\right|_{\mu}$.

By applying Eq. 7 and Eq. 8 to Eq. 10, finally the motion bound can be obtained as Eq. 11 where $\left|{}^{j-1}\mathbf{L}_j(t)\right|_{\mu}$ can be calculated as Eq. 7 or Eq. 8.

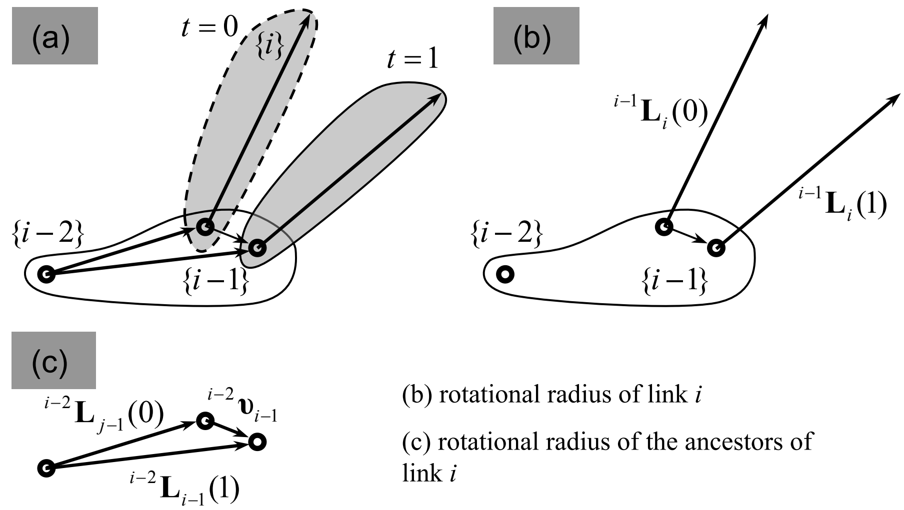

Figure 1: **Rotational Radius.** ***(a)*** *A link i undergoes a motion with constant translational and rotational velocities with respect to its parent's frame $\{i-1\}$ during the time interval $[0,1]$.* ***(b)*** *The rotational radius of a point* $\mathbf{p}$ *on the current link i is defined as a vector from* $\mathbf{p}$ *to its rotational center (e.g. the origin of its parent's frame).* ***(c)*** *In contrast, the rotational radius of the ancestor link j of the current link i ($j > i$) is defined as a vector from the origin of the frame j to the origin of its parent's frame as shown in (c).*

3 Effects of Link Motion on the Motion Bound

We can rewrite Eq. 11 as Eq. 12 to show the contribution of individual link motion to the final motion bound. When other factors such as $\mathbf{n}, {}^{j-1}\mathbf{L}_j$ remain constant, roughly speaking, the motion of jth link, ${}^{j-1}\mathbf{v}_j, {}^{j-1}\boldsymbol{\omega}_j$, contributes to the motion bound of ith link ($j < i$) linearly dependent of their height difference (i.e., $i - j$).

$$
\begin{aligned}
& \max_{{}^0_i\mathbf{p}\in\mathscr{A}_i} \int_0^1 \left|{}^0\mathbf{v}_i \cdot \mathbf{n}\right| dt \\
=\ & \max_{{}^0_i\mathbf{p}\in\mathscr{A}_i} \int_0^1 \left| \left(\sum_{j=1}^{i-1} \left({}^0_{j-1}\mathbf{R}^{j-1}\mathbf{v}_j + \left(\sum_{k=1}^{j} {}^0_{k-1}\mathbf{R}^{k-1}\omega_k \right) \times {}^0_{j-1}\mathbf{R}^{j-1}\mathbf{L}_j \right) \right) \cdot \mathbf{n} \right| dt \qquad (9) \\
=\ & \max_{{}^0_i\mathbf{p}\in\mathscr{A}_i} \int_0^1 \left| \left({}^0\mathbf{v}_1 + {}^0\omega_1 \times {}^0\mathbf{L}_1 + \sum_{j=2}^{i-1} \left({}^0_{j-1}\mathbf{R}^{j-1}\mathbf{v}_j + \left(\sum_{k=1}^{j} {}^0_{k-1}\mathbf{R}^{k-1}\omega_k \right) \times {}^0_{j-1}\mathbf{R}^{j-1}\mathbf{L}_j \right) \right) \cdot \mathbf{n} \right| dt \\
=\ & \max_{{}^0_i\mathbf{p}\in\mathscr{A}_i} \int_0^1 \left| \left({}^0\mathbf{v}_1 \cdot \mathbf{n} + {}^0\omega_1 \times {}^0\mathbf{L}_1 \cdot \mathbf{n} + \sum_{j=2}^{i-1} \left({}^0_{j-1}\mathbf{R}^{j-1}\mathbf{v}_j \cdot \mathbf{n} + \left(\sum_{k=1}^{j} {}^0_{k-1}\mathbf{R}^{k-1}\omega_k \right) \times {}^0_{j-1}\mathbf{R}^{j-1}\mathbf{L}_j \cdot \mathbf{n} \right) \right) \right| dt \\
\le\ & \max_{{}^0_i\mathbf{p}\in\mathscr{A}_i} \int_0^1 \left(\left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|{}^0\omega_1 \times {}^0\mathbf{L}_1 \cdot \mathbf{n}\right| + \sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \mathbf{n} \times \left(\sum_{k=1}^{j} {}^0_{k-1}\mathbf{R}^{k-1}\omega_k \right) \cdot {}^0_{j-1}\mathbf{R}^{j-1}\mathbf{L}_j \right) \right) dt \\
\le\ & \max_{{}^0_i\mathbf{p}\in\mathscr{A}_i} \int_0^1 \left(\left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|\mathbf{n} \times {}^0\omega_1 \cdot {}^0\mathbf{L}_1\right| + \sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \left| \mathbf{n} \times \left({}^0\omega_1 + \sum_{k=2}^{j} {}^0_{k-1}\mathbf{R}^{k-1}\omega_k \right) \right| \left|{}^{j-1}\mathbf{L}_j\right| \right) \right) dt \\
\le\ & \max_{{}^0_i\mathbf{p}\in\mathscr{A}_i} \int_0^1 \left(\left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|\mathbf{n} \times {}^0\omega_1\right| \left|{}^0\mathbf{L}_1\right| + \sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \left(\left|\mathbf{n} \times {}^0\omega_1\right| + \sum_{k=2}^{j} \left|{}^{k-1}\omega_k\right| \right) \left|{}^{j-1}\mathbf{L}_j\right| \right) \right) dt \qquad (10) \\
\le\ & \int_0^1 \left(\left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|\mathbf{n} \times {}^0\omega_1\right| \left|{}^0\mathbf{L}_1(t)\right|_\mu + \sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \left(\left|\mathbf{n} \times {}^0\omega_1\right| + \sum_{k=2}^{j} \left|{}^{k-1}\omega_k\right| \right) \left|{}^{j-1}\mathbf{L}_j(t)\right|_\mu \right) \right) dt \\
=\ & \left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|\mathbf{n} \times {}^0\omega_1\right| \left|{}^0\mathbf{L}_1(t)\right|_\mu + \sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \left(\left|\mathbf{n} \times {}^0\omega_1\right| + \sum_{k=2}^{j} \left|{}^{k-1}\omega_k\right| \right) \left|{}^{j-1}\mathbf{L}_j(t)\right|_\mu \right) \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
& \left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|\mathbf{n} \times {}^0\omega_1\right| \left|{}^0\mathbf{L}_1\right|_\mu + \sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \left(\left|\mathbf{n} \times {}^0\omega_1\right| + \sum_{k=2}^{j} \left|{}^{k-1}\omega_k\right| \right) \left|{}^{j-1}\mathbf{L}_j\right|_\mu \right) \\
=\ & \left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|\mathbf{n} \times {}^0\omega_1\right| \left|{}^0\mathbf{L}_1\right|_\mu + \left|\mathbf{n} \times {}^0\omega_1\right| \sum_{j=2}^{i-1} \left|{}^{j-1}\mathbf{L}_j\right|_\mu + \sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \left(\sum_{k=2}^{j} \left|{}^{k-1}\omega_k\right| \right) \left|{}^{j-1}\mathbf{L}_j\right|_\mu \right) \\
=\ & \underbrace{\left|{}^0\mathbf{v}_1 \cdot \mathbf{n}\right| + \left|\mathbf{n} \times {}^0\omega_1\right| \sum_{j=1}^{i-1} \left|{}^{j-1}\mathbf{L}_j\right|_\mu}_{\mathrm{root}(j=1)} + \underbrace{\sum_{j=2}^{i-1} \left(\left|{}^{j-1}\mathbf{v}_j\right| + \left|{}^{j-1}\omega_j\right| \left(\sum_{k=j}^{i-1} \left|{}^{k-1}\mathbf{L}_k\right|_\mu \right) \right)}_{\mathrm{other-links}(j=2,3,\cdots,i-1)} \qquad (12)
\end{aligned}
$$