Analysis of Non-stationary Interpolatory Subdivision Schemes Based on Exponential Polynomials

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Abstract: In this study, we are concerned with non-stationary interpolatory subdivision schemes with refinement rules which may vary from level to level. First, we derive a new class of interpolatory non-stationary subdivision schemes reproducing exponential polynomials. Next, we prove that non-stationary schemes based on the known butterfly-shaped stencils possess the same smoothness as the known Butterfly interpolatory scheme.

1 Introduction

A non-stationary subdivision scheme consists of recursive refinements of an initial sparse sequence with the use of rules that may vary from level to level but are the same everywhere on the same level. Our primary concern of this study to provide a non-stationary subdivision scheme which is exact on a certain shift-invariant space which consists of trigonometric polynomials.

In this study, instead of thinking about generating new vertices in $\mathbb{R}^3$ by taking local averages of vertices in the control polyhedron, we are thinking of scalar values which are assigned to the vertices of a triangulation in $\mathbb{R}^2$. Further, we are particulary interested in the class of interpolatory subdivision schemes which refine data by inserting values corresponding to intermediate points, using linear combinations of neighboring points. The general form of their refinement rules (based on a planar parametric domain) is as follows:

$$\begin{aligned} f_{2j}^{k+1} &= f_j^k, \\ f_{2j+1}^{k+1} &= \sum_{n\in\mathbb{Z}^2} a_{j-2n}^{[k]} f_n^k, \quad j\in\mathbb{Z}^2,\ k\in\mathbb{Z}_+. \end{aligned}$$

The set of coefficients $a^{[k]} := \{a_n^{[k]}\}$ is termed the mask of the rule at level k. We denote this rule by $S_{a^{[k]}}$ and the corresponding non-stationary scheme by $\{S_{a^{[k]}}\}$. It is common to assume that for each level k, only a finite number of coefficients $a_n^{[k]}$ are non-zero so that changes in a control point affect only its local neighborhood. This property clearly facilitates the practical implementation. A subdivision scheme is said to be stationary when the mask is independent of the levels; then we use the notation $a := \{a_n\}$. We denote the rule at each level by S_a and the corresponding stationary scheme by $\{S_a\}$.

Let f^0 be a given initial data. For a non-stationary subdivision scheme $\{S_{a^{[k]}}\}$, we have the formal relation

$$f^k = S_{a^{[k-1]}} \cdots S_{a^{[0]}} f^0.$$

In particular, for the given data $\delta = \{\delta_{n,0} : n \in \mathbb{Z}^2\}$ at level 0, with the Kronecker delta $\delta_{n,0}$, the *basic limit function* of $\{S_{a^{[k]}}\}$ is the function

$$\phi_0 := \lim_{k\to\infty} S_{a^{[k]}} \cdots S_{a^{[0]}} \delta.$$

Definition 1.1. *A subdivision scheme is said to be C^ν if for the initial data $\delta = \{f_n^0 = \delta_{n,0} : n \in \mathbb{Z}^2\}$, there exists a limit function $\phi_0 \in C^\nu(\mathbb{R})$, $\phi_0 \not\equiv 0$, satisfying*

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}^2} |f_n^k - \phi_0(2^{-k}n)| = 0. \tag{1}$$

Let $\{S_{a^{[k]}}\}$ be a non-stationary interpolatory subdivision scheme based on the butterfly-shaped stencils. Assume that $\{S_{a^{[k]}}\}$ reproduces exponential polynomials which constitute a shift-invariant space $\mathbb{S}$ with $\#\mathbb{S} = 8$. Under certain condition of $\mathbb{S}$, we prove that $\{S_{a^{[k]}}\}$ is asymptotically equivalent to the original Butterfly scheme $\{S_a\}$, and that $\{S_{a^{[k]}}\}$ is C^1, i.e., it has the same smoothness as the stationary Butterfly subdivision scheme $\{S_a\}$.

In the following we use the two-dimensional index notation. First, let

$$\mathbb{Z}_+^2 := \{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_i \ge 0,\ i = 1,2\},$$

Let $\alpha, \beta \in \mathbb{Z}_+^2$. We denote by D^α the differential operator of order α, and $|\alpha|_1 := \alpha_1 + \alpha_2$. Also, $\alpha \le \beta$ means $\alpha_i \le \beta_i$ for $i = 1,2$. For $x = (x_1, x_2) \in \mathbb{R}^2$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$. The space of algebraic polynomials of degree $< n$ is denoted by $\Pi_{<n}$. For a matrix $\mathbf{C}$, $\|\mathbf{C}\|_\infty$ indicates its ∞-norm.

2 Construction of Non-stationary Schemes

The non-stationary subdivision scheme proposed in this paper are constructed in a way of reproducing the exponential (or trigonometric) polynomials of the form

$$\phi(u,v) = e^{\beta_1 u} e^{\beta_2 v}, \quad (u,v) \in \Omega \subset \mathbb{R}^2, \tag{2}$$

with complex numbers β_i $(i = 1,2)$. One may employ more general type of exponential polynomials

$$\phi(u,v) = u^{\alpha_1} v^{\alpha_2} e^{\beta_1 u} e^{\beta_2 v},$$

$\alpha_i = 0,\cdots,\mu \in \mathbb{Z}_+$, with (larger) stencils. However, the same technique of this study can be applied for the analysis. In practice, u^{α_i}'s are chosen to be low degree polynomials.

Let

$$\mathbb{S} = \{\phi_\ell(u,v) \,|\, \ell = 1,\cdots,8\} \tag{3}$$

be a shift-invariant space with linearly independent ϕ_ℓ's of the form as in (2). In fact, refining the triangular mesh data based on the butterfly-shaped stencils involves three groups of vertices to evaluate three types of new vertices as shown in Figure 1. Here and in the sequel, $\mathscr{X}_j$ with $j = 1,2,3$ indicate the three types of stencils and $p_j/2$ the insertion points corresponding to $\mathscr{X}_j$ at level 0. For each level $k = 0,1,\cdots$, and stencil $\mathscr{X}_j$, the non-stationary subdivision rule is constructed by solving the linear system

$$\phi_\ell(p_j 2^{-k-1}) = \sum_{n\in\mathscr{X}_j} a^{[k]}_{p_j-2n} \phi_\ell(n2^{-k}), \quad \phi_\ell \in \mathbb{S}. \tag{4}$$

This linear system can be written in the matrix form

$$\mathbf{a}^{[k]} = \mathbf{B}^{[k]^{-1}} \mathbf{b}^{[k]}$$

where $\mathbf{a}^{[k]} = (a^{[k]}_{p_j-2n} : n \in \mathscr{X}_j)$, $\mathbf{B}^{[k]}(\ell,n) = (\phi_\ell(n2^{-k}) : n \in \mathscr{X}_j,\ \ell = 1,\cdots,8)$ and $\mathbf{b}^{[k]} = (\phi_\ell(p_j 2^{-k-1}) : \ell = 1,\cdots,8)$. For the unique solution of the linear system (4), it is required that $\dim(\mathbb{S}|_{\mathscr{X}_j}) = \dim\mathbb{S}$ for each stencil $\mathscr{X}_j$. This condition is satisfied by the basic assumption: With T_{φ_ℓ} the Taylor polynomial around zero of φ_ℓ of degree < 4, i.e.,

$$T_{\varphi_\ell}(\cdot) = \sum_{|\nu|_1 \le 3} (\cdot)^\nu \varphi_\ell^{(\nu)}(0)/\nu!, \tag{5}$$

the functions T_{φ_ℓ}, $\ell = 1,\cdots,8$, are linearly independent.

3 Asymptotic Equivalence

Typical tools used for the analysis of the stationary schemes are Fourier transform, eigen analysis and Laurent polynomials [1, 2]. However, these techniques are not

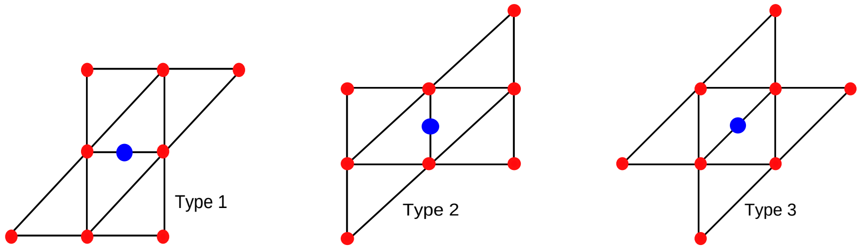

Figure 1. Stencils of the Butterfly scheme. Blue dots indicate the insertion point.

applicable for the non-stationary case. Thus, for the analysis of non-stationary schemes, we adopt the notion of asymptotical equivalence [5]: two (non-stationary) subdivision schemes $\{S_{a_k}\}$ and $\{S_{\bar{a}^{[k]}}\}$ are *asymptotically equivalent*, if

$$\sum_{k\in\mathbb{Z}_+} \|S_{a^{[k]}} - S_{\bar{a}^{[k]}}\|_\infty < \infty, \tag{6}$$

where

$$\|S_{a^{[k]}}\|_\infty = \max\left\{\sum_{n\in\mathbb{Z}^2} |a^{[k]}_{\alpha-2n}| : \alpha \in E^2\right\}$$

where E^2 stands for the extreme points of $[0,1]^2$.

4 Analysis of Convergence

We cite a basic result about the asymptotically equivalent subdivision schemes.

Theorem 4.1. ([5]) *Let $\{S_a\}$ be a stationary subdivision scheme, and let $\{S_{a^{[k]}}\}$ and $\{S_a\}$ be asymptotically equivalent. Assume that $supp\{a_n\}, supp\{a^{[k]}_n\} \subset [0,N]^2$, $k \in \mathbb{Z}_+$, $N < \infty$. Then $\{S_{a^{[k]}}\}$ is C^0 if $\{S_a\}$ is C^0. Moreover, if*

$$\|S_{a^{[k]}} - S_a\|_\infty \le c2^{-k}, \quad k \in \mathbb{Z}_+,$$

then the basic limit function ϕ_0 of the scheme $\{S_{a^{[k]}}\}$ is Hölder continuous of exponent $\nu > 0$.

In the following theorem, we show that the non-stationary subdivision scheme $\{S_{a^{[k]}}\}$ is asymptotically equivalent to the Butterfly scheme $\{S_a\}$, which implies that $\{S_{a^{[k]}}\}$ is a convergent scheme. Recall that the mask $\{a_{p_j-2n} : n \in \mathscr{X}_j\}$ of the Butterfly scheme reproduce polynomials in $\Pi_{<4}$ in the sense that

$$\sum_{n\in\mathscr{X}_j} a_{p_j-2n} p(n2^{-k}) = p(2^{-k-1}p_j), \quad \forall p \in \Pi_{<4}. \tag{7}$$

Theorem 4.2. *Let $\{a_n\}$ be the mask of the Butterfly scheme $\{S_a\}$ and $\{a_n^{[k]}\}$ be the mask of the non-stationary scheme $\{S_{a^{[k]}}\}$ reproducing the space $\mathbb{S} = \{\varphi_\ell : \ell = 1, \cdots, 8\}$. Then, there exists a constant $c > 0$ such that*

$$\max_{n\in\mathbb{Z}} |a_n^{[k]} - a_n| \le c2^{-k}, \quad k \ge K \in \mathbb{Z}_+.$$

Proof. Due to the fact that $a_{2n} = a_{2n}^{[k]} = \delta_{n,0}$ with $n \in \mathbb{Z}^2$, it remains to estimate only the difference $a_{p_j-2n} - a_{p_j-2n}^{[k]}$ with $p_1 = (1,0)$, $p_2 = (0,1)$ and $p_3 = (1,1)$.

Let T_{φ_ℓ} be the Taylor polynomial of degree 3 of the function φ_ℓ, $\ell = 1, \cdots, 8$, as in (5) and let $j = 1,2,3$ be fixed. Since the mask $\{a_{p_j-2n}\}$, $n \in \mathscr{X}_j$, of the Butterfly scheme reproduce polynomials in $\Pi_{<4}$ as in (7), we have

$$\sum_{n\in\mathscr{X}_j} a_{p_j-2n} T_{\varphi_\ell}(n2^{-k}) = T_{\varphi_\ell}(2^{-k-1}p_j).$$

This linear system can be written in the matrix form

$$\mathbf{T} \cdot \mathbf{a} = \mathbf{b} \tag{8}$$

with

$$\mathbf{T} = (T_{\varphi_\ell}(n2^{-k})), \;\; \mathbf{a} = (a_{p_j-2n}), \;\; \mathbf{b} = (T_{\varphi_\ell}(2^{-k-1}p_j)).$$

Recall that the mask $\{a_{j-2n}^{[k]}\}$ of the non-stationary scheme reproduces φ_ℓ in the sense that

$$\sum_{n\in\mathscr{X}_j} a_{p_j-2n}^{[k]} \varphi_\ell(n2^{-k}) = \varphi_\ell(2^{-k-1}p_j). \tag{9}$$

Let $R_{\varphi_\ell} := \varphi_\ell - T_{\varphi_\ell}$ be the remainder of the Taylor polynomial, i.e.,

$$R_{\varphi_\ell}(\cdot) = \sum_{|\alpha|_1=4} \frac{(\cdot)^\alpha}{\alpha!} \int_0^1 (1-y)^3 \varphi_\ell^{(\alpha)}(y\cdot)dy. \tag{10}$$

It is obvious that $|R_{\varphi_\ell}(n2^{-k})| \le c2^{-4k}$, $n \in \mathscr{X}_j$. Thus, denoting $\mathbf{a}^{[k]} = (a_{p_j-2n}^{[k]} : n \in \mathscr{X}_j)$ and $\mathbf{R} = (2^{4k}R_{\varphi_\ell}(n2^{-k}) : n \in \mathscr{X}_j)$, the linear system (9) is of the form

$$(\mathbf{T} + \varepsilon\mathbf{R})\mathbf{a}^{[k]} = \mathbf{b} + \varepsilon\mathbf{b}_{\mathbf{R}}$$

with $\varepsilon = 2^{-4k}$, $\|\mathbf{R}\|_\infty < \infty$ and $\|\mathbf{b_R}\|_\infty < \infty$. Note that the $O(\varepsilon)$ perturbation of the non-singular matrix $\mathbf{T}$ causes the $O(\varepsilon)$ perturbation of its inverse [6]. Thus, we can write

$$\begin{aligned}\mathbf{a}_k^{[k]} &= (\mathbf{T}^{-1} + \varepsilon\tilde{\mathbf{R}})(\mathbf{b} + \varepsilon\mathbf{b_R}) \\ &= \mathbf{a} + \varepsilon[\mathbf{T}^{-1}\mathbf{b_R} + \tilde{\mathbf{R}}(\mathbf{b} + \varepsilon\mathbf{b_R})],\end{aligned}$$

with $\|\tilde{\mathbf{R}}\|_\infty < \infty$. Since $\varepsilon = 2^{-4k}$ and $\|\mathbf{T}^{-1}\|_\infty = O(2^{3k})$, we conclude that there is a constant $c > 0$ such that $\|\mathbf{a}_k^{[k]} - \mathbf{a}\|_\infty \leq c2^{-k}$. □

5 Smoothness of Non-stationary Scheme

5.1 Sufficient Condition for Smoothness

In this section, we infer results on the smoothness of the non-stationary interpolatory scheme $\{S_{a^{[k]}}\}$ from known results about the smoothness of a stationary Butterfly scheme, which is asymptotically equivalent to it. Specifically, we will prove that the non-stationary scheme reproducing the space $\mathbb{S}$ has the same smoothness as the original Butterfly scheme which reproduces the cubic polynomials.

To simplify the presentation of a subdivision scheme and its analysis, it is convenient to assign the Laurent polynomial to each subdivision mask. The bivariate Laurent polynomial corresponding to the original Butterfly scheme can be put in factored form [2]

$$a(z) = a(z_1, z_2) = 2^{-1}(1 + z_1^{-1})(1 + z_2^{-1})(1 + z_1^{-1}z_2^{-1})z_1z_2(1 - 16^{-1}q(z_1, z_2)), \quad (11)$$

where

$$\begin{aligned}q(z) = q(z_1, z_2) \quad &= \quad 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1} \\ &\quad +2z_1^{-1}z_2 + 2z_1z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2.\end{aligned}$$

It seems that the Laurent polynomial $a(z_1, z_2)$ has single roots along the curves $z_1 = -1$ $z_2 = -1$ and $z_1z_2 = -1$. However, $a(z_1, z_2)$ has multiple roots around $(-1, 1)$ and (1, -1). In the following, for convenience, we use the notation

$$e_1 = (1, 0) \quad \text{and} \quad e_2 = (0, 1). \quad (12)$$

Lemma 5.1. *Let $a(z) = \sum_{n \in \mathbb{Z}^2} a_n z^n$ be the Laurent polynomials of the original Butterfly subdivision scheme $\{S_a\}$. Then, for any $\beta_j = me_j$, $j = 1, 2$, with $m \leq 3$ and e_j in (12), we have*

$$D^{\beta_j} a(\theta_j) = 0$$

where

$$\theta_1 = (-1,1) \quad \text{and} \quad \theta_2 = (1,-1). \tag{13}$$

Proof. In this proof, we consider only the case $j=1$ because the analysis for the case $j=2$ is exactly same. For $\beta_j = me_j$ with $m \le 3$, we can write

$$D^{\beta_j} a(-1,1) = \sum_{|\ell|_1 \le |\beta_j|_1} \gamma_{\beta_j,\ell} \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} a_n n^\ell (-1)^{n_1}$$

for some suitable constants $\gamma_{\beta_j,\ell}$. From the fact that the mask $\{a_n\}$ reproduces polynomials of degree 3, it is easy to see that

$$\sum_{n=(n_1,n_2)\in\mathbb{Z}^2} a_n n^\ell (-1)^{n_1} = \delta_{\ell,0} - \sum_{j=1}^{3} (-1)^j \sum_{n\in\mathscr{X}_j} a_{p_j-2n}(p_j/2-n)^\ell = 0$$

The lemma is proved. □

When the non-stationary subdivision scheme $\{S_{a^{[k]}}\}$ is asymptotically equivalent to the original Butterfly scheme $\{S_a\}$, $|a_n^{[k]} - a_n| = o(1)$, as k tends to ∞. Hence, the Laurent polynomial $a^{[k]}(z)$ associated with $S_{a^{[k]}}$ has (complex) roots around $z_1=-1$, $z_2=-1$ and $z_1z_2=-1$, and it can be written as

$$\begin{aligned} a^{[k]}(z) &:= a^{[k]}(z_1,z_2) \\ &:= 2^{-1}(1+w_{1,k}z_1)(1+w_{2,k}z_2)((1+w_{3,k}z_1z_2)c^{[k]}(z_1,z_2) \end{aligned} \tag{14}$$

for some suitable Laurent polynomial $c^{[k]}(z)$. From Lemma 5.1, it is easy to see that for $\nu = 0,\cdots,3$ and $j=1,2$, $\frac{\partial^\nu}{\partial z_j^\nu} a^{[k]}(\theta_j) = o(1)$ with θ_j in (13) as $k\to\infty$. In this study, we require the stronger conditions:

Condition A. *A non-stationary subdivision scheme $\{S_{a^{[k]}}\}$ satisfies Condition A if the corresponding Laurent polynomials $a^{[k]}(z)$ are of the form* (14) *and if*

$$\Big|\frac{\partial^\nu}{\partial z_j^\nu} a^{[k]}(\theta_j)\Big| \le c2^{-(4-\nu)k}, \quad j=1,2, \quad \nu=0,\cdots,3,$$

with θ_j in (13).

In what follows, we prove that $\{S_{a^{[k]}}\}$ with Laurent polynomials of the form (14) satisfying Condition A has the smoothness C^1 if $\{S_{b^{[k]}}\}$ is C^0. For this, we show that a factor $(1+r_kz^\lambda)$, $\lambda\in\mathbb{Z}^2$, in the Laurent polynomials of $a^{[k]}(z)$ with $|1-r_k| \le c2^{-k}$ is a smoothing factor.

Lemma 5.2. *Consider a non-stationary subdivision scheme* $\{S_{a^{[k]}}\}$ *with Laurent polynomials of the form*

$$a^{[k]}(z) = \frac{1}{2}(1 + r_k z^{\lambda}) b^{[k]}(z), \quad k > K \in \mathbb{Z}_+$$

where $\lambda \in \mathbb{Z}^2$ *and* $\{S_{b^{[k]}}\}$ *is* C^0*. Suppose that*

$$|1 - r_k| \leq c2^{-k}, \quad k \geq K \in \mathbb{Z}_+, \tag{15}$$

and that the scheme $\{S_{b^{[k]}}\}$ *is of compact support and* C^{γ}*. Then the basic limit function* $\phi_0 = \lim_{k\to\infty} S_{a^{[k]}} \cdots S_{a_0}\delta$ *satisfies* $\phi_0, \partial_\lambda \phi_0 \in C^0$ *where* $\delta = \{\delta_{n,0} : n \in \mathbb{Z}^2\}$*.*

Remark: For the Butterfly scheme in regular triangulation, we need to consider only the three directions: $\lambda = (1,0), (0,1), (1,1)$.

Proof. Due to Lemma 9 in [5], we can find that $\{S_{a^{[k]}}\}$ is C^0 with the basic limit function ϕ_a of the form

$$\phi_0 = \int_{\mathbb{R}} \phi_b(\cdot - \lambda t) h(t) dt,$$

where ϕ_b and h are the basic limit functions of $\{S_{b^{[k]}}\}$ and $\{S_{1+r_k z}\}$ respectively, and where h is bounded and $\text{supp}\{h\} = [0,1)$. Then, it follows that $\phi_0 \in C^0$. In order to prove $\partial_\lambda \phi_0 \in C^0$, it is sufficient to show that $\phi_0 = \phi_b * h \in C^1(\mathbb{R})$ under the assumption that ϕ_b is continuous. To this end, let us define the sequence of functions

$$I_k(x) = \int_{x-1}^{x} \phi_b(t) h_k(x-t)\, dt,$$

where

$$h_k(t) = h(j2^{-k}), \quad j2^{-k} \leq t < (j+1)2^{-k}, \quad j = 0, \cdots, 2^k - 1. \tag{16}$$

It follows that $I_k(x) \to \phi_a(x)$ uniformly as $k \to \infty$. To establish that $\phi_a \in C^1(\mathbb{R})$, let us examine the sequence of functions $\{I'_k\}_{k\in\mathbb{Z}_+}$. By (16),

$$I'_k(x) = \sum_{j=0}^{2^k-1} h(j2^{-k})[\phi_b(x - j2^{-k}) - \phi_b(x - (j+1)2^{-k})] \in C(\mathbb{R}),$$

and hence,

$$\begin{aligned} I'_{k+1}(x) - I'_k(x) &= \sum_{j=0}^{2^k-1} [h((j+2^{-1})2^{-k}) - h(j2^{-k})] \\ &\quad \cdot [\phi_b(x - (j+2^{-1})2^{-k}) - \phi_b(x - (j+1)2^{-k})]. \end{aligned}$$

Here, since ϕ_b is Hölder continuous of exponent $\nu > 0$ (see Theorem 4.1),

$$|\phi_b(x-(j+2^{-1})2^{-k}) - \phi_b(x-(j+1)2^{-k})| \leq c2^{-\nu k}$$

with a constant $c > 0$ independent of j and x. In addition, seeing that the Laurent polynomial corresponding to h is $1 + r_k z$, we find that

$$h((j+\frac{1}{2})2^{-k}) = r_k h(j2^{-k});$$

see Example 2 in [5] for the details. Thus, we obtain the expression

$$|I'_{k+1}(x) - I'_k(x)| \leq c2^{-\nu k} \sum_{j=0}^{2^k-1} |(r_k - 1)h(j2^{-k})| \leq c' 2^{-\nu k},$$

being a consequence of $|1 - r_k| \leq c2^{-k}$ in (15) and the boundedness of h. Accordingly, we conclude that $\{I'_k\}$ is uniformly convergent, and hence the limit is continuous and is the derivative of $\phi_a = \phi_b * h$. □

5.2 Analysis of Smoothness

It is known that the subdivision scheme corresponding to the Laurent polynomial

$$b_j(z_1, z_2) = 2(1+z_j)^{-1} a(z_1, z_2), \quad j = 1, 2,$$

with $a(z_1, z_2)$ in (11) is C^0 [4]. Since the non-stationary scheme $\{S_{a^{[k]}}\}$ is asymptotically equivalent to the Butterfly scheme $\{S_a\}$, the scheme corresponding to

$$b_j^{[k]}(z_1, z_2) = 2(1 + w_{j,k} z_j)^{-1} a^{[k]}(z_1, z_2), \quad j = 1, 2, \tag{17}$$

with $a^{[k]}(z_1, z_2)$ in (14) is also C^0. Thus, according to Lemma 5.2, the C^1-smoothness of the non-stationary scheme $\{S_{a^{[k]}}\}$ can be proved by showing that $|1 - w_{j,k}| \leq c2^{-k}$, $j = 1, 2$. Specifically, we show that Condition A on $\{S_{a^{[k]}}\}$ implies the condition (15) for all the factors in the representation (14). For this proof, we use the notation

$$\{x_k\} \asymp \{y_k\}$$

if there exist some constants $c_1, c_2 > 0$ such that $c_1 \leq x_k y_k^{-1} \leq c_2$ for all k. (Here, $y_k \neq 0$.)

Lemma 5.3. *Suppose that Condition A holds for the non-stationary subdivision scheme $\{S_{a^{[k]}}\}$. Let $b_j^{[k]}(z_1, z_2)$ in (17) be the Laurent polynomial of $\{S_{a^{[k]}}\}$ (14). Then, for each $j = 1, 2$,*

$$|1 - w_{j,k}| \leq c2^{-k}, \quad k \geq K \in \mathbb{Z}_+. \tag{18}$$

Proof. Since $\{S_{a^{[k]}}\}$ is asymptotically equivalent to $\{S_a\}$, we deduce from Lemma 5.1 that $a^{[k]}(z_1,1)$ and $a^{[k]}(1,z_2)$ can be written as follows:

$$\bar{a}_{1,k}(z_1) := a^{[k]}(z_1,1) = \bar{c}_{1,k}(z_1)\prod_{n=1}^{4}(1+r_{k,n}z_1), \tag{19}$$
$$\bar{a}_{2,k}(z_2) := a^{[k]}(1,z_2) = \bar{c}_{2,k}(z_2)\prod_{n=1}^{4}(1+s_{k,n}z_2),$$

where $\bar{c}_{j,k}(-1) = c + o(1)$, $j = 1,2$, and $c \neq 0$. We will show that $|1-r_{k,n}|, |1-s_{k,n}| \leq c2^{-k}$ for $k \geq K \in \mathbb{Z}_+$ for any $n = 1,\cdots,4$. It is enough for $w_{j,k}$ to be $|1-w_{j,k}| \leq c2^{-k}$, $k \geq K \in \mathbb{Z}_+$, $j = 1,2$. For this, it is also necessary to point out that

$$\Big(\frac{\partial^\nu}{\partial z_j^\nu} a^{[k]}\Big)(\theta_j) = \Big(\frac{d^\nu}{d z_j^\nu}\bar{a}_{1,k}\Big)(-1) \tag{20}$$

with θ_j in (13).

Without loss of generality, we rearrange the set $r_{k,n}$ in (14) such that

$$|1-r_{k,n}| = \max\{|1-r_{k,\ell}| : 4 \geq \ell \geq n\}, \quad n = 1,\cdots,4, \tag{21}$$

that is, $|1-r_{k,n}| \geq |1-r_{k,n+1}|$. Denote $|1-r_{k,1}| =: \varepsilon_k$. Since $|1-r_{k,n}| \leq \varepsilon_k$ for any $n \leq 4$, it is sufficient to show that $\sup_k |2^k\varepsilon_k| \leq c$ for a constant $c > 0$. Now, suppose that $\sup_k |2^k\varepsilon_k| = \infty$, which means that there exists a sequence $\{k_\ell\}$ such that

$$|2^{k_\ell}\varepsilon_{k_\ell}| \leq |2^{k_{\ell+1}}\varepsilon_{k_{\ell+1}}| \to \infty, \quad \text{as} \quad k_\ell \to \infty. \tag{22}$$

Then, recalling $|1-r_{k,n+1}| \leq |1-r_{k,n}|$, we will derive a contradiction by considering the following two cases:

Case 1: $\{\varepsilon_{k_\ell}\} \asymp \{|1-r_{k_\ell,n}|\}$, for $n = 1,\cdots,4$.

In this case, it is clear from (14) that $\{|\bar{a}_{1,k_\ell}(-1)|\} \asymp \{\varepsilon_{k_\ell}^4\}$. By Condition A and (20), $|\bar{a}_{1,k_\ell}(-1)| \leq c2^{-k_\ell 4}$, we get the bound $|2^{k_\ell}\varepsilon_{k_\ell}| \leq c$ for any k_ℓ, in contradiction to (22).

Case 2: $\{\varepsilon_{k_\ell}\} \asymp \{|1-r_{k_\ell,n}|\}$, for $n = 1,\cdots,s < 4$.

That is, there exists a subsequence $\{k_j\} \subset \{k_\ell\}$ such that for any $n > s$, $|1-r_{k_j,n}|\varepsilon_{k_j}^{-1} \to 0$ as $k_j \to \infty$, i.e.,

$$|1-r_{k_j,n}| = o(\varepsilon_{k_j}), \quad n > s. \tag{23}$$

Then we use the lemma:

Lemma 5.4. *Let*

$$F_{k_j}(z_1) := \prod_{n=1}^{4} \frac{1}{2}(1 + r_{k_j,n} z_1).$$

Under the condition of Case 2, we have

$$\{|F_{k_j}^{(4-s)}(-1)|\} \asymp \{\varepsilon_{k_j}^s\} \quad \text{and} \quad |F_{k_j}^{(4-s-\ell)}(-1)| = o(\varepsilon_{k_j}^{s+\ell}), \quad \forall \ell > 0.$$

Proof. For the given $s < 4$, denote $I_s := \{1, \cdots, s\}$ and let Λ_s be the collection of all subsets of $\{1,2,3,4\} =: I_4$ with the cardinality s, i.e., $\Lambda_s := \{I \subset I_4 \;:\; \#I = s\}$. Then,

$$F_{k_j}^{(4-s)}(-1) \;=\; \left(\prod_{n\in I_s} (1 - r_{k_j,n}) + \sum_{I\in\Lambda_s\setminus I_s} \prod_{n\in I} (1 - r_{k_j,n}) \right) \left(\frac{1}{2^4} + o(1) \right). \quad (24)$$

Since $|1 - r_{k_j,n}| \geq |1 - r_{k_j,n+1}|$,

$$\left\{ \prod_{n\in I_s} |1 - r_{k_j,n}| \right\} \asymp \{\varepsilon_{k_j}^s\} \quad \text{and} \quad \prod_{n\in I} |1 - r_{k_j,n}| = o(\varepsilon_{k_j}^s).$$

Thus,

$$\{|F_{k_j}^{(4-s)}(-1)|\} \asymp \{\varepsilon_{k_j}^s\}.$$

In a similar way, we can prove the relation $|F_{k_j}^{(4-s-\ell)}(-1)| = o(\varepsilon_{k_j}^{s+\ell})$ for all $\ell > 0$. □

Now, we turn to the proof of Lemma 5.3 in Case 2. It follows from (19) that for some suitable constants c_ℓ with $\ell = 0, \cdots, 4-s$, we have

$$\begin{aligned} \bar{a}_{1,k_j}^{(4-s)}(-1) &= \sum_{\ell=0}^{4-s} \binom{4-s}{\ell} \bar{c}_{1,k_j}^{(\ell)}(-1) F_{k_j}^{(4-s-\ell)}(-1) \qquad (25) \\ &= \bar{c}_{1,k_j}(-1) F_{k_j}^{(4-s)}(-1) + \sum_{\ell=1}^{4-s} \binom{4-s}{\ell} \bar{c}_{1,k_j}^{(\ell)}(-1) F_{k_j}^{(4-s-\ell)}(-1). \end{aligned}$$

Since $\bar{c}_{1,k_j}(-1) = c + o(1)$ with a constant $c \neq 0$, the identity (25) leads to

$$\{\bar{a}_{1,k_j}^{(4-s)}(-1)\} \asymp \{\varepsilon_{k_j}^s\}$$

by Lemma 5.4. Also, from Condition A and (20), $|\bar{a}_{1,k_j}^{(4-s)}(-1)| \leq c2^{-k_j s}$, yielding $|2^{k_j} \varepsilon_{k_j}| \leq c$ for any k_j, in a contradiction to (22). (Here c is a generic constant). Therefore, we can conclude that $|1 - w_{1,k}| \leq c2^{-k}$.

we can prove that $|1-\overline{r}_{k,n}| \le c2^{-k}$. Therefore, we conclude (18). □

Recall that $\mathbb{S} = \{\phi_\ell(x) : \ell = 1, \cdots, 8\}$ be the shift-invariant space reproduced by the non-stationary scheme $\{S_{a^{[k]}}\}$. Let

$$\mathbb{T} = \{\nu \in \mathbb{Z}_+^2 : |\nu|_1 \le 3,\ \nu \ne (1,2),(2,1)\}$$

and let $\beta = me_j,\ j = 1,2$, with $m \le 3$ and e_j in (12). Here, note that $\#\mathbb{T} = 8$. Then, for each stencil $\mathscr{X}_j$ with $j = 1,2,3$, let us define the function

$$\Phi_{\beta,j}(x) := \Phi^{[k]}_{\beta,j}(x) := \sum_{\ell=1}^{8} g^{[k]}_{\beta,j}(\ell)\phi_\ell(x) \tag{26}$$

so that the coefficients $g^{[k]}_{\beta,j}(\ell)$, $\ell = 1, \cdots, 8$, is obtained by solving the linear system

$$\Phi^{(\nu)}_{\beta,j}(2^{-k-1}p_j) = \delta_{\beta,\nu}(-1)^{\nu}\nu! \tag{27}$$

where $p_1 = (1,0),\ p_2 = (0,1)$ and $p_3 = (1,1)$. This linear system (27) can be written in the matrix form

$$\mathbf{P}_k \cdot \mathbf{g}^{[k]}_{\beta} = \mathbf{c}$$

with

$$\mathbf{P}_k := \big(\phi^{(\nu)}_\ell(p_j 2^{-k-1}) : \nu \in \mathbb{T},\ \phi_\ell \in \mathbb{S}\big)$$
$$\mathbf{c} := \big(\delta_{\beta,\nu}(-1)^{|\nu|_1}\nu! : \nu \in \mathbb{T}\big).$$

The non-singularity of $\mathbf{P}_k$ is clear from the condition of $\mathbb{S}$. Then it is necessary to point that since the mask $\{a^{[k]}_n\}$ reproduces ϕ_ℓ with $\ell = 1, \cdots, 8$, it also reproduces $\Phi_{\beta,j}$ in the following sense:

$$\Phi_{\beta,j}(2^{-k-1}p_j) = \sum_{n\in\mathscr{X}_j} a^{[k]}_{p_j-2n}\Phi_{\beta,j}(n2^{-k}), \quad j = 1,2,3. \tag{28}$$

The following lemma is necessary for the proof of Condition A.

Lemma 5.5. *Let $\Phi_{\beta,j}$ be the function defined as in (26) and let $\beta = me_j,\ j = 1,2$, with $m \le 3$ and e_j in (12). Then, for any $\nu \in \mathbb{Z}_+^2$ with $|\nu|_1 \le 3$,*

$$\|\Phi^{(\nu)}_{\beta,j}\|_{L_\infty[-\eta,\eta]} = O(2^{3k}), \quad \text{as } k \to \infty, \tag{29}$$

with $\eta > 0$.

Proof. Since $\mathbf{g}_\beta^{[k]} = \mathbf{P}_k^{-1} \cdot \mathbf{c}$, the estimate $\|\mathbf{g}_\beta^{[k]}\|_\infty \le \|\mathbf{P}_k^{-1}\|_\infty \|\mathbf{c}\|_\infty$. Using the same technique in Theorem 4.2, we can prove $\|\mathbf{P}_k^{-1}\|_\infty = O(2^{3k})$ with $k \ge K$. Then the lemma follows immediately. □.

We are now ready to prove the main theorem of this section which actually proves that the Laurent polynomial $a^{[k]}(z)$ of the scheme $\{S_{a^{[k]}}\}$ satisfies the Condition A.

Theorem 5.6. *Let $a^{[k]}(z) = \sum_{n\in\mathbb{Z}^2} a_n^{[k]} z^n$ be the Laurent polynomial at level k associated with the non-stationary interpolatory scheme $\{S_{a^{[k]}}\}$. Then, for any $\beta_j = me_j \in \mathbb{Z}_+^2$ with $m \le 3$, we have*

$$|D^{\beta_j} a^{[k]}(\theta_j)| \le c2^{-k(4-|\beta_j|)}, \quad k \ge K \in \mathbb{Z}_+$$

with θ_j in (13).

Proof. In this proof, we consider only the case $j=1$ because the case $j=2$ can be done by exactly the same way. Seeing that

$$D^{\beta_1} a^{[k]}(-1,1) = \sum_{|\ell|_1 \le |\beta_1|_1} \gamma_{\beta_1,\ell} \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} a_n^{[k]} n^\ell (-1)^{n_1}$$

for some constants $\gamma_{\beta_1,\ell}$, it is sufficient to show that for any $|\beta_1|_1 \le 3$,

$$s_{\beta_1} := \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} (-1)^{n_1} n^{\beta_1} a_n^{[k]} = O(2^{-k(4-|\beta_1|_1)}), \quad k \to \infty,$$

in order to conclude Condition A. Note that $a_{2n}^{[k]} = \delta_{n,0}$ because $\{a_n^{[k]}\}$ is the mask of an interpolatory scheme. Hence,

$$2^{-|\beta_1|_1(k+1)} s_{\beta_1} = \delta_{\beta_1,0} - \sum_{j=1}^{3} (-1)^j \sum_{n\in\mathscr{X}_j} a_{p_j-2n}^{[k]} \big((p_j/2-n)2^{-k}\big)^{\beta_1} \tag{30}$$

where $\mathscr{X}_j$ is the butterfly stencil of type $j = 1,2,3$ (see Figure 1) and $p_j/2$ is the corresponding insertion point. Invoking (26) and (28), we get

$$\delta_{\beta_1,0} = \Phi_{\beta_1,j}(2^{-k-1}p_j) = \sum_{n\in\mathscr{X}_j} a_{p_j-2n}^{[k]} \Phi_{\beta_1,j}(n2^{-k}).$$

This together with (30) lead to

$$2^{-|\beta_1|_1(k+1)} s_{\beta_1} = \sum_{j=1}^{3} (-1)^{j-1} \sum_{\ell\in\mathscr{X}_j} a_{p_j-2n}^{[k]} \left(\Phi_{\beta_1,j}(n2^{-k}) - ((p_j/2-n)2^{-k})^{\beta_1} \right). \tag{31}$$

Here, we replace $\Phi_{\beta_1,j}(n2^{-k})$ by its Taylor polynomial of degree up to 3 plus the remainder term. The Taylor expansion of $\Phi_{\beta_1,j}$ around $2^{-k-1}p_j$ of degree up to 3 is

$$\Phi_{\beta_1,j}(n2^{-k}) = T^{[k]}_{\beta_1,j}(n2^{-k}) + R^{[k]}_{\beta_1,j}(n2^{-k})$$

where the Taylor polynomial $T^{[k]}_{\beta_1,j}$ and the remainder $R^{[k]}_{\beta_1,j}$ are given by

$$T^{[k]}_{\beta_1,j}(n2^{-k}) = \sum_{|\nu|_1\leq 3} \left((n-p_j/2)2^{-k}\right)^{\nu} \Phi^{(\nu)}_{\beta_1,j}(2^{-k-1}p_j)/\nu!$$
$$R^{[k]}_{\beta_1,j}(n2^{-k}) = \sum_{|\nu|_1=4} \left((n-p_j/2)2^{-k}\right)^{\nu} \Phi^{(\nu)}_{\beta_1,j}(\xi 2^{-k})/\nu!$$

with ξ a point between $p_j 2^{-k-1}$ and $n2^{-k}$. Invoking (27), i.e., $\Phi^{(\nu)}_{\beta_1,j}(2^{-k-1}p_j) = \delta_{\beta_1,\nu}(-1)^{\nu}\nu!$ with $\nu \in \mathbb{T}$, we see that

$$T^{[k]}_{\beta_1,j}(n2^{-k}) - ((p_j/2-n)2^{-k})^{\beta_1} = \sum_{\nu=(1,2),(2,1)} ((n-p_j/2)2^{-k})^{\nu} \Phi^{(\nu)}_{\beta_1,j}(2^{-k-1}p_j)/\nu!$$

Applying Lemma 4.2 and the polynomial reproducing property of the mask $\{a_n\}$ of the Butterfly scheme (7), it is immediate that for each $\nu = (1,2),(2,1)$,

$$|\sum_{\ell\in\mathscr{X}_j} a^{[k]}_{p_j-2n}((n-p_j/2)2^{-k})^{\nu}| \leq c2^{-4k} \tag{32}$$

for a constant $c > 0$. Also, it is easy to check that $|\sum_{\ell\in\mathscr{X}_j} a^{[k]}_{p_j-2n} R^{[k]}_{\beta_1,j}(n2^{-k})| \leq c2^{-4k}$.This together with (31) and (32), we obtain the required result. □

We are now ready to provide the main theorem of this section.

Theorem 5.7. *If $\{S_{a^{[k]}}\}$ is the non-stationary interpolatory subdivision scheme reproducing the shift-invariant space $\mathbb{S}$, then $\{S_{a^{[k]}}\}$ is C^1, i.e., it has the same smoothness as the stationary Butterfly subdivision scheme $\{S_a\}$.*

Proof. From Lemmas 5.2, 5.3, and Theorem 5.6, the proof is immediate. □

6 Approximation Order

An important issue in the implementation of subdivision algorithm is how to actually attain the original function as close as possible if the given initial data f^0 is sampled from an underlying function. A high quality reconstruction scheme should guarantee that the approximation error decreases when the quality of the sample increases. This approximation power is usually quantified by the approximation order. For simplicity, suppose that f^0 is of the form $f^0 := \{f^0_n = f(2^{-k_0}n) : n \in \mathbb{Z}^2\}$ for some positive integer k_0 with a smooth function f. Then our concern is to find to the largest exponent $m > 0$ such that

$$\|f^\infty - f\|_{L_\infty(K)} \le C2^{-mk_0}$$

with a constant $C > 0$ independent of k_0, where K is a compact set in $\mathbb{R}^2$. The exponent m is called the *approximation order* of the subdivision scheme. It is well-known that the original Butterfly scheme reproduces cubic polynomials and hence it provides the approximation order 4. The goal of this section is to show that the non-stationary scheme $\{S_{a^{[k]}}\}$ provides the same approximation order 4 as the original Butterfly scheme.

Since the scheme $\{S_{a^{[k]}}\}$ is uniformly convergent, its limit function can be written as

$$\lim_{\ell\to\infty} S_{a^{[k_0+\ell]}} \cdots S_{a^{[k_0]}} f^0 = \sum_{n\in\mathbb{Z}^2} \psi_{k_0}(2^{k_0}\cdot -n) f(2^{-k_0}n) \tag{33}$$

where ψ_{k_0} is the basic limit function of S defined by

$$\psi_{k_0} = \lim_{\ell\to\infty} S_{a^{[k_0+\ell]}} \cdots S_{a^{[k_0]}} \delta$$

with $\delta = \{\delta_{0,n}\}$. Indeed, since $\{S_{a^{[k]}}\}$ reproduces functions in $\mathbb{S}$, it is clear that

$$\phi_\ell = \sum_{n\in\mathbb{Z}^2} \phi_\ell(2^{-k_0}n)\psi_{k_0}(2^{k_0}\cdot -n), \quad \forall \phi_\ell \in \mathbb{S}.$$

This observation leads to the proof of the approximation order of the subdivision scheme $\{S_{a^{[k]}}\}$. In this paper, we are particularly interested in approximating functions f in the Sobolev space

$$W^\gamma\infty(K) = \{g : \sum_{|\alpha|_1\le\gamma} \|D^{(\alpha)}g\|_{L_\infty(K)} < \infty\}, \quad \gamma \in \mathbb{Z}_+.$$

Theorem 6.1. *Let K be a compact set in $\mathbb{R}^2$. Assume that $f \in W^4_\infty(K)$ and $f^0 := \{f^0_n = f(2^{-k_0}n) : n \in \mathbb{Z}^2\}$. Then the non-stationary interpolatory scheme $\{S_{a^{[k]}}\}$ has the approximation order 4 with respect to functions in $W^4_\infty(K)$.*

Proof. Recall the notation $\mathbb{T} = \{\nu \in \mathbb{Z}^2_+ : |\nu|_1 \le 3,\ \nu \ne (1,2),(2,1)\}$, let us define the function

$$\Phi(x) := \sum_{\ell=1}^{8} \mu_\ell \phi_\ell(x)$$

so that the coefficient vector $\mathbf{m}_x := \{\mu_\ell : \ell = 1, \cdots, 8\}$ is obtained by solving the linear system

$$D^\nu \Phi(x) = D^\nu f(x)$$

Since Φ is a linear combination of ϕ_ℓ, $\ell = 1, \cdots, 8$, and the scheme $\{S_{a^{[k]}}\}$ is exact for such functions, we easily get the identity

$$\Phi(x) = \sum_{\ell \in \mathbb{Z}} \psi_{k_0}(2^{k_0} - n)\Phi(n2^{-k}).$$

Then it follows that

$$\begin{aligned} f(x) - f^\infty(x) &= \Phi(x) - \sum_{n \in \mathbb{Z}^2} \psi_{k_0}(2^{k_0}x - n) f(n2^{-k_0}) \\ &= \sum_{n \in \mathbb{Z}^2} \psi_{k_0}(2^{k_0}x - n)\big(\Phi(n2^{-k_0}) - f(n2^{-k_0})\big) \end{aligned}$$

Denote by T_g the Taylor polynomial of a smooth function g of degree 3 around x, i.e.,

$$T_g(y) = \sum_{|\nu|_1 \le 3} (y - x)^\nu g^{(\nu)}(x)/\nu!$$

and let R_g be its remainder

$$R_g(y) = \sum_{|\nu|_1 = 4} (y - x)^\nu g^{(\nu)}(\xi)/\nu!.$$

for some ξ between x and y. Then, due to the fact $\Phi^{(\nu)}(x) = f^{(\nu)}(x)$ with $\nu \in \mathbb{T}$,

$$T_\Phi(n2^{-k_0}) - T_f(n2^{-k_0}) = \sum_{\nu = (1,2),(2,1)} \frac{(n2^{-k_0} - x)^\nu}{\nu!}(\Phi - f)^{(\nu)}(x).$$

It follows that

$$\begin{aligned} F(x) - f^\infty(x) &= \sum_{n \in \mathbb{Z}^2} \psi_{k_0}(2^{k_0}x - n) \sum_{\nu = (1,2),(2,1)} \frac{(n2^{-k_0} - x)^\nu}{\nu!}(\Phi - f)^{(\nu)}(x) \\ &\quad + \sum_{n \in \mathbb{Z}^2} \psi_{k_0}(2^{k_0}x - n)(R_\Phi(n2^{-k_0}) - R_f(n2^{-k_0})). \end{aligned} \tag{34}$$

Letting ψ be the basic limit function of the Butterfly scheme, i.e., $\psi = S^\infty \delta$, we find from Lemma 15 in [5] that

$$\|\psi_{k_0} - \psi\|_\infty \le c2^{-k_0}. \tag{35}$$

Applying the polynomial reproducing property of the Butterfly scheme (7), it is immediate that for each $\nu = (1,2),(2,1)$,

$$\sum_{n\in\mathbb{Z}^2} \psi(2^{k_0}x-n)(n2^{-k_0}-x)^{\nu} = 0.$$

Note that since ψ_{k_0} is compactly supported, $\#\{n\in\mathbb{Z}^2 : \psi_{k_0}(2^{k_0}-n)\neq 0\} \leq C_{\psi_{k_0}}$ for any x. Hence, using (35), it is easy to check that the first term on the right-hand side of (34) is $O(2^{-k_0 4})$. Also, since Φ is bounded on any compact set K, it is obvious the second term has the property $O(2^{-k_0 4})$. It finishes the proof. □

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Appendix

In the appendix, we show parametric equations, basis functions(B), shift-invariant spaces ($\mathbb{S}$), and stencils ($\mathscr{X}$) for different surfaces that were used as benchmarking models in our work.

1. Sphere

$$\begin{aligned} x(u,v) &= r\sin u\cos v, \\ y(u,v) &= r\sin u\sin v, \\ z(u,v) &= r\cos u, \end{aligned}$$

where $0 \leq v \leq 2\pi, 0 \leq u \leq \pi$ and some constant r.

$$\begin{aligned} B &:= \{\sin u\cos v, \sin u\sin v, \cos u\} \\ \mathbb{S} &:= \{\sin u, \cos u, \sin v, \cos v, \\ &\quad \sin u\sin v, \cos u\cos v, \sin u\cos v, \cos u\sin v\} \\ \mathscr{X} &:= \text{Fig. 1} \end{aligned}$$

2. Torus

$$\begin{aligned} x(u,v) &= (1+\cos u)\cos v, \\ y(u,v) &= (1+\cos u)\sin v, \\ z(u,v) &= \sin u, \end{aligned}$$

where $0 \leq u, v \leq 2\pi$.

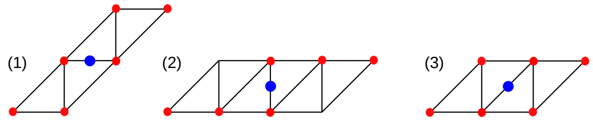
$$\begin{aligned} B &:= \{\cos u\cos v, \cos u\sin v, \sin u\} \\ \mathbb{S} &:= \{\sin u, \cos u, \sin v, \cos v, \\ &\quad \sin u\sin v, \cos u\cos v, \sin u\cos v, \cos u\sin v\} \\ \mathscr{X} &:= \text{Fig. 1} \end{aligned}$$

3. Möbius Strip

$$\begin{aligned} x(u,v) &= a\cos u + v\cos(u/2), \\ y(u,v) &= a\sin u + v\cos(u/2), \\ z(u,v) &= v\sin(u/2), \end{aligned}$$

where $0 \leq u \leq 2\pi$, $-w \leq v \leq w$ and w and a are constants.

$$\begin{aligned} B &:= \{\sin u, \cos u, v\sin(u/2), v\cos(u/2)\}, \\ \mathbb{S} &:= \{\sin u, \cos u, \sin(u/2), \cos(u/2), v\sin(u/2), v\cos(u/2)\}, \\ \mathscr{X} &:= \end{aligned}$$

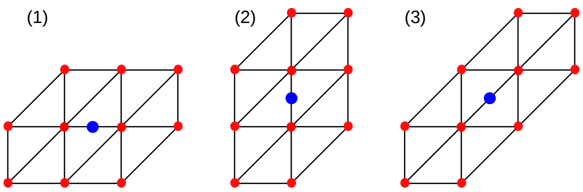


4. Fish surface

$$x(u,v) = (\cos u - \cos(2u))\cos v/4,$$
$$y(u,v) = (\sin u - \sin(2u))\sin v/4,$$
$$z(u,v) = \cos u,$$

where $0 \le u \le \pi$, $0 \le v \le 2\pi$.

$$\begin{aligned} B &:= \{\cos(2u)\cos v, \sin(2u)\sin v, \cos u \\ &\quad \cos u \cos v, \sin u \sin v\} \\ \mathbb{S} &:= \{\sin u \cos v, \sin u \sin v, \cos u \sin v, \cos u \cos v, \\ &\quad \sin(2u)\cos v, \sin(2u)\sin v, \cos(2u)\sin v, \\ &\quad \cos(2u)\cos v, \cos u, \sin u\} \\ \mathscr{X} &:= \end{aligned}$$

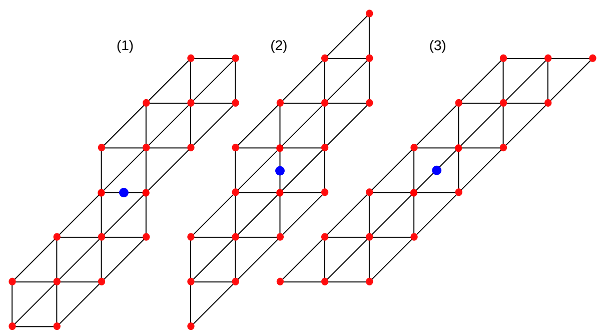


5. Sea shell surface

$$x(u,v) = a(1-\frac{v}{2\pi})\cos(nv)(1+\cos u) + c\cos(nv),$$
$$y(u,v) = a(1-\frac{v}{2\pi})\sin(nv)(1+\cos u) + c\sin(nv),$$
$$z(u,v) = b\frac{v}{2\pi} + a(1-\frac{v}{2\pi})\sin u,$$

where $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ and a, b, c, and n are constants.

$$
\begin{aligned}
B &:= \{\, v, \sin u, v\sin u, \cos(3v), \sin(3v), v\cos(3v) \\
&\quad v\sin(3v), v\cos(3v)\cos u, v\sin(3v)\cos u \} \\
\mathbb{S} &:= \{\, 1, v, \sin u, \cos u, v\sin(3v), v\cos(3v), \\
&\quad \cos(3v), \sin(3v), v\cos(u), v\sin(u), \\
&\quad v\cos(3v)\cos u, v\cos(3v)\sin u, v\sin(3v)\cos u, \\
&\quad v\sin(3v)\sin u, \sin(3v)\sin u, \sin(3v)\cos u, \\
&\quad \cos(3v)\cos u, \cos(3v)\sin u \} \\
\mathscr{X} &:=
\end{aligned}
$$

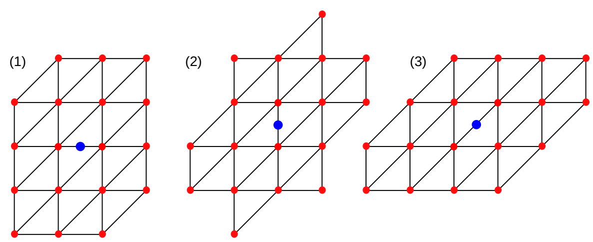


6. Figure 8 klein bottle

$$
\begin{aligned}
x(u,v) &= (a + \cos(u/2)\sin v - sin(u/2)\sin(2v))\cos u, \\
y(u,v) &= (a + \cos(u/2)\sin v - sin(u/2)\sin(2v))\sin u, \\
z(u,v) &= \sin(u/2)\sin v + \cos(u/2)\sin(2v),
\end{aligned}
$$

where $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ and some constant a.

$$\begin{aligned}
B &:= \{ \cos(u/2)\sin v\cos u, \sin(u/2)\cos(2v)\sin u, \\
&\quad \cos(u/2)\sin v\sin u, \sin(u/2)\sin(2v)\sin u, \\
&\quad \cos(u/2)\sin(2v), \sin(u/2)\sin v\} \\
\mathbb{S} &:= \{ \sin u, \cos u, \sin(u/2)\sin v, \\
&\quad \sin(u/2)\cos v, \cos(u/2)\sin v, \cos(u/2)\cos v, \\
&\quad \sin(u/2)\sin(2v), \sin(u/2)\cos(2v), \\
&\quad \cos(u/2)\sin(2v), \cos(u/2)\cos(2v), \\
&\quad \sin(3u/2)\sin v, \sin(3u/2)\cos v, \\
&\quad \cos(3u/2)\sin v, \cos(3u/2)\cos v, \\
&\quad \sin(3u/2)\sin(2v), \sin(3u/2)\cos(2v), \\
&\quad \cos(3u/2)\sin(2v), \cos(3u/2)\cos(2v)\} \\
\mathscr{X} &:=
\end{aligned}$$

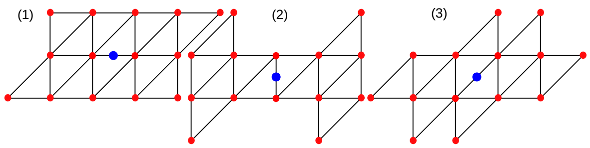


7. Klein bottle

$$\begin{aligned}
r(u,v) &= 4(1-\cos u), \\
x(u,v) &= 6\cos u(1+\sin u) + r\cos u\cos v, \quad 0 \le u < \pi, \\
&\quad 6\cos u(1+\sin u) + r\cos(v+\pi), \quad \pi \le u < 2\pi, \\
y(u,v) &= 16\sin u + r\sin u\cos v, \quad 0 \le u < \pi, \\
&\quad 16\sin u, \quad \pi \le u < 2\pi, \\
z(u,v) &= r\sin v,
\end{aligned}$$

where $0 \le v \le 2\pi$.

$$\begin{aligned} B &:= \{ \cos u, \sin u, \cos u \sin u, \cos v, \\ &\quad \cos^2 u \cos v, \cos u \cos v \} \\ \mathbb{S} &:= \{ \cos u, \sin u, \cos(2u), \sin(2u), \cos v, \sin v, \\ &\quad \sin u \sin v, \sin u \cos v, \cos u \cos v, \cos u \sin v, \\ &\quad \sin(2u) \sin v, \sin(2u) \cos v, \cos(2u) \cos v, \\ &\quad \cos(2u) \sin v \} \\ \mathscr{X} &:= \end{aligned}$$



8. Superellipsoid

$$\begin{aligned} x(u,v) &= r_x \cos^{n_1} u \cos^{n_2} v, \\ y(u,v) &= r_y \cos^{n_1} u \sin^{n_2} v, \\ z(u,v) &= r_z \sin^{n_1} u, \end{aligned}$$

where $-\pi/2 \le u \le \pi/2$, $-\pi \le v \le \pi$, $0 < n_1, n_2 < \infty$ and r_x, r_y, r_z are constants.

In case $(n_1, n_2) = (3, 1)$

$$\begin{aligned} B &:= \{ \cos^3 u \sin v, \cos^3 u \sin v, \sin^3 u \} \\ \mathbb{S} &:= \{ \cos(3u), \sin(3u), \cos u, \sin u, \sin u \sin v, \\ &\quad \sin u \cos v, \cos u \cos v, \cos u \sin v, \sin(3u) \sin v, \\ &\quad \sin(3u) \cos v, \cos(3u) \cos v, \cos(3u) \sin v \} \\ \mathscr{X} &:= \end{aligned}$$